Segmentation as a variational method

Level-Sets

Outline

1. Image Segmentation
   - Geodesic Active Contours
   - Level Set Method
   - Region-based formulation
   - Image Classification

Segmentation

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The process of partitioning the image support Ω into disjoint regions $R_i \in \Omega$, where $\bigcup_i R_i = \Omega$.

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Goal

Splitting image into objects and background?
Splitting image into regions of different intensities?
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In all the cases, edges are important features

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Evolution of explicit curve

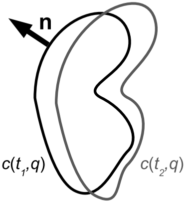
- Family of closed curves in time and space:

$$C = \{c(t,q) : R^+ \times \langle 0,1 \rangle \to \Omega, c(t,0) = c(t,1)\}$$

$$c(t,q) = [c_1(t,q), c_2(t,q)]$$

$$\mathbf{t} = c' = [\tfrac{\partial c_1}{\partial q}, \tfrac{\partial c_2}{\partial q}]$$ tangent direction

$$\mathbf{n} = (c')^{\perp}/|c'|$$ unit normal vector



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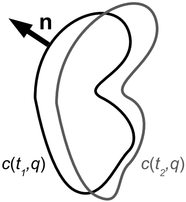


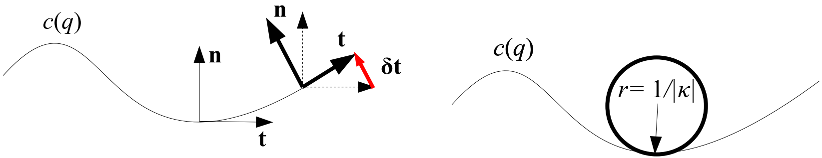
- Motion in the normal direction (evolution equation)

$$\frac{\partial c}{\partial t} = F\mathbf{n},$$

where F is “speed” function.

Curvature

$$\frac{\partial \mathbf{t}}{\partial q} = \kappa \frac{\mathbf{n}}{|\mathbf{n}|}$$



$$\kappa = \frac{c_1' c_2'' - c_2' c_1''}{((c_1')^2 + (c_2')^2)^{3/2}}$$

Mean curvature motion

$$\frac{\partial c}{\partial t} = F\mathbf{n} = \kappa\mathbf{n}$$

Play avi

Snakes

Kass, Witkin, Terzopolous '88

$$J(c) = \underbrace{\int_0^1 |c'(q)|^2 + \beta |c''(q)|^2 dq}_{\text{internal energy}} + \lambda \underbrace{\int_0^1 g^2(|\nabla I(c(q))|)\, dq}_{\text{external energy}}$$

Image $I : \Omega \to R$, parametric curve $c : \langle 0, 1\rangle \to \Omega$

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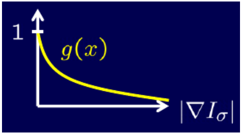
- Edge-detector function

$$g(x) = \frac{1}{1 + x^2}$$



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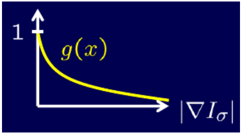

- The model is too complicated!!!
- Change parametrization of $c \Rightarrow$ different solution.

Geodesic active contours

Caselles, Kimmel, Sapiro '97

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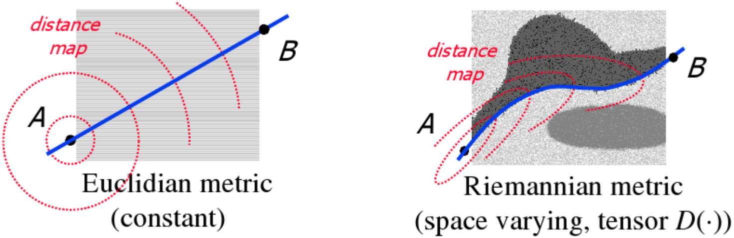
- Equivalent to "Snakes" but far more simple
- Compute geodesics: shortest curve between two points Riemannian metric from the image gradient



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κg:

- mean curvature motion weighted by g
- stop evolving the curve when at the object boundary

$g \equiv 1$

Play avi

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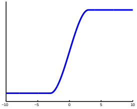
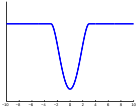
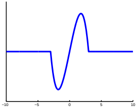
$\nabla g \cdot \mathbf{n}$:

- increase the attraction towards the object boundary

I 

g 

g' 

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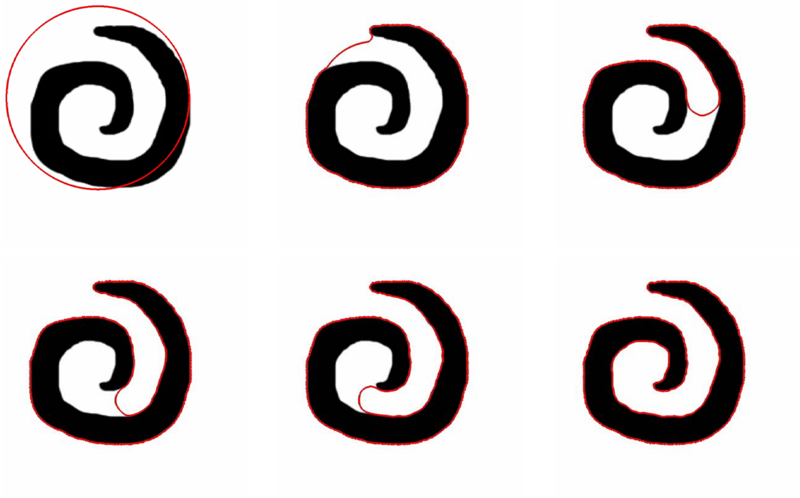
P.D.E. (evolution equation):

$$\frac{\partial c}{\partial t} = (\kappa g - \nabla g \cdot \mathbf{n} + \alpha g)\mathbf{n}$$

αg:

- increases the speed of convergence,
- makes detection of nonconvex objects easier, $\alpha + \kappa$ must remain of constant sign,
- deduced from area energy: $\int_{\text{inside}(c)} g \, dxdy$

Evolution of explicit boundaries



Cremers, Tischhäuser, Weickert, Schnörr, "Diffusion Snakes", IJCV '02

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- Difficult numerical approximation

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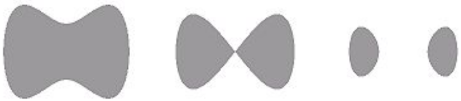
- Cannot change topology



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- Solution: level set method

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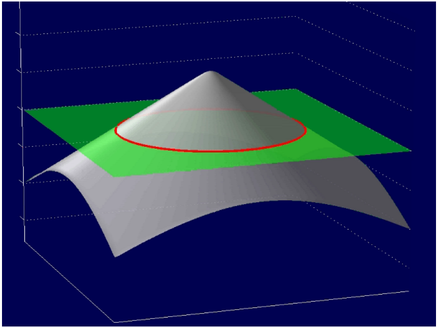

$$\phi(x) : \Omega \rightarrow R \,, \quad c = \{x \in \Omega | \phi(x) = 0\}$$

Osher, Sethian, J. of Comp. Phys. '88

A curve can be seen as the zero-level of a function in higher dimension.

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- Assume the curve evolves according to: $\frac{\partial c}{\partial t} = F\mathbf{n}$

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- Since $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$, we obtain the level set equation:

$$\frac{\partial \phi}{\partial t} = -F|\nabla\phi|$$

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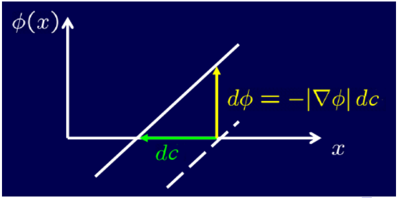
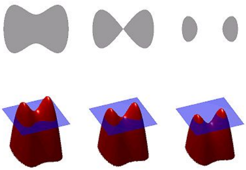
- The scaling by $|\nabla \phi|$ is easily verified in one dimension:



Pros and Cons

+ May change topology



Wikipedia

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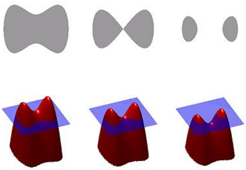

Wikipedia

+ Easy numerical approximation: finite-difference approx. for the spatial and temporal variables

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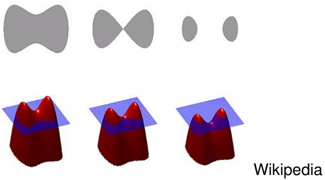

+ Easy numerical approximation: finite-difference approx. for the spatial and temporal variables
+ Intrinsic geometry elements easily expressed:

$$\mathbf{n} = \nabla\phi / |\nabla\phi| \qquad \text{normal vector}$$

$$\kappa = -\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) \qquad \text{curvature}$$

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  Surface . . . zero-level set of a function defined in a volume.

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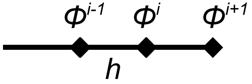
  - symmetric differences, upwind schemes, . . .



$$\phi_x^i \approx \frac{\phi^{i+1} - \phi^i}{h}, \quad \phi_x^i \approx \frac{\phi^i - \phi^{i-1}}{h}, \quad \phi_x^i \approx \frac{\phi^{i+1} - \phi^{i-1}}{2h}$$

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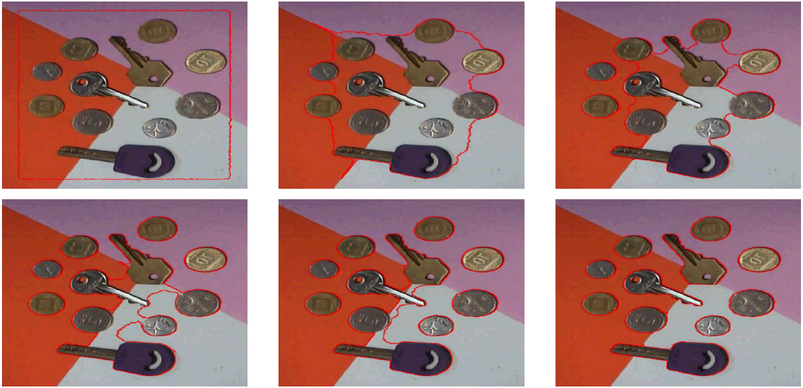
Level set formulation:

$$\frac{\partial \phi}{\partial t} = -\left((\kappa + \alpha)g - \nabla g \cdot \frac{\nabla \phi}{|\nabla \phi|}\right)|\nabla \phi|$$

$$\frac{\partial \phi}{\partial t} = g(|\nabla I|)\left(\text{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) - \alpha\right)|\nabla \phi| + \nabla g \cdot \nabla \phi$$

Example



Goldenberg, Kimmel, Rivlin, Rudzsky, IEEE TIP '01

Observation

$$J(c) = \int_0^1 |c'(q)|dq$$

$$\downarrow$$

$$\frac{\partial c}{\partial t} = \kappa \mathbf{n} \qquad \longleftrightarrow \qquad \frac{\partial \phi}{\partial t} = \text{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)|\nabla \phi|$$

Observation

$$J(c) = \int_0^1 |c'(q)|dq \quad \longleftrightarrow \quad F(\phi) = \int_\Omega |\nabla\phi| dx\,, \quad |\nabla\phi| = 1$$

$$\downarrow \qquad\qquad\qquad\qquad \uparrow \quad \text{E-L eq.}$$

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Mumford-Shah functional

Mumford, Shah '89

$$F(u,K) = \int_{\Omega \setminus K} (u - I)^2 dx + \alpha \int_{\Omega \setminus K} |\nabla u|^2 dx + \beta \int_K ds$$

- $\Omega \in R^2$ ...image domain
- $I : \Omega \to R$ ...input image
- $u : \Omega \to R$ ...segmented image
- $K \in \Omega$ set of discontinuities

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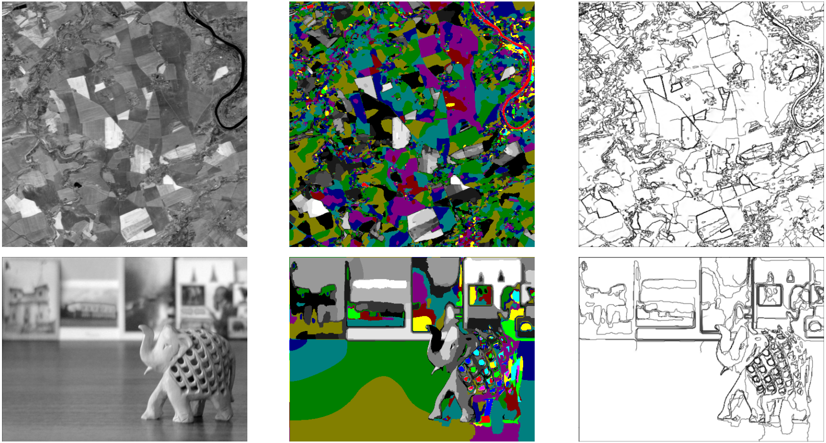
- Difficulty: involves two unknowns u and K.
- Many ways how to approximate M-S and eliminate K.
- And again we solve the corresponding E-L equation.

Example



I: input image u: segmented image K: discontinuity set

Reduced Mumford-Shah functional

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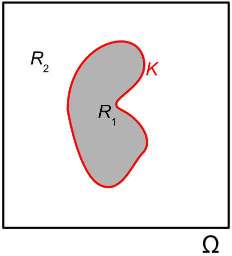
- $\alpha \to \infty$: piecewise constant model
- $K \in C$: two-region problem



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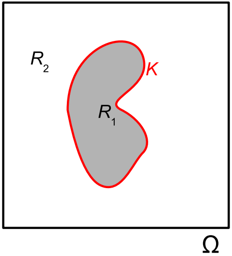

$$F(u,K) = \int_{R_1} |u_1 - I|^2 dx + \int_{R_2} |u_2 - I|^2 dx + \beta |K|$$

u_i...mean in R_i

F becomes function of *K* only.

Level set formulation of Mumford-Shah

Chan, Vese '01

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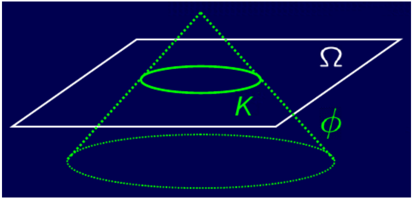

$$H(\phi) = \begin{cases} 1, & \text{if } \phi > 0 \\ 0, & \text{else} \end{cases}$$

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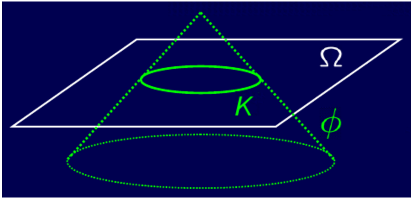

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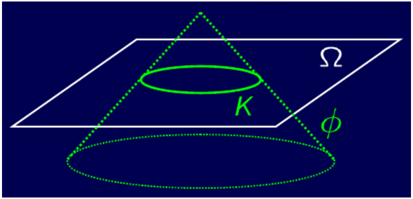

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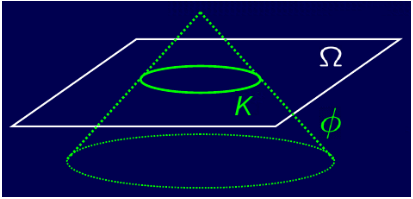

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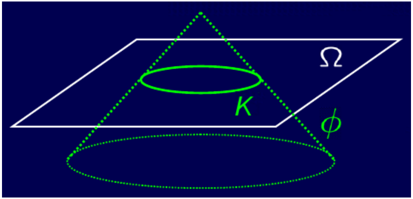

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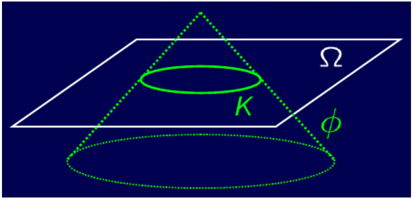


$$H(\phi) = \begin{cases} 1, & \text{if } \phi > 0 \\ 0, & \text{else} \end{cases}$$

use smoothed step function

$$F(u_1, u_2, \phi) = \int_\Omega |I - u_1|^2 H\phi + |I - u_2|^2 (1 - H\phi) dx + \beta \int_\Omega |\nabla H\phi| dx$$

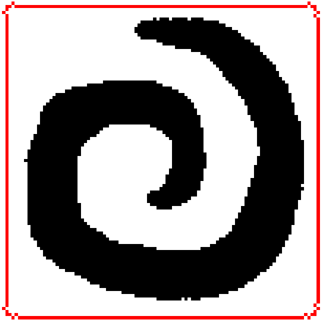
$$\frac{\partial \phi}{\partial t} = -F'(\phi) = \delta(\phi) \left(\beta \, \text{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) - |I - u_1|^2 + |I - u_2|^2 \right)$$

Edge-based x Region-based

$$\frac{\partial \phi}{\partial t} = g(|\nabla I|)\left(\operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) - \alpha\right)|\nabla \phi| + \nabla g \cdot \nabla \phi$$

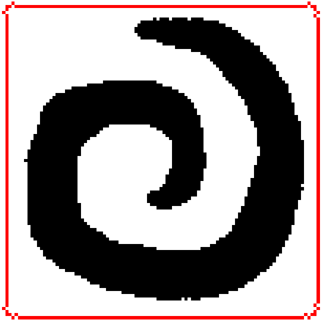
$$\frac{\partial \phi}{\partial t} = \delta(\phi)\left(\beta \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) - |I - u_1|^2 + |I - u_2|^2\right)$$

Examples



Play avi

Edge-based



Play avi

Region-based

Examples

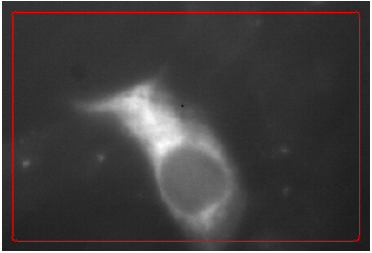
Play avi

Edge-based

Play avi

Region-based

Examples



Play avi

Outline

1. Image Segmentation
   - Geodesic Active Contours
   - Level Set Method
   - Region-based formulation
   - Image Classification

Classification

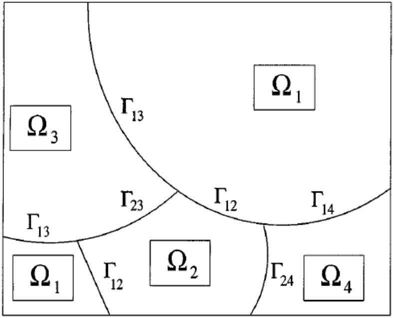


image I

- Classes $1, \ldots, K$ and we know for each class (μ_i, σ_i).
- Each region Ω_i has its corresponding level set ϕ_i.

Classification

- Three conditions: partitioning, interclass similarity, length shortening

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- $$F^C(\phi_1,\ldots,\phi_K) = \sum_{i=1}^K \int_\Omega |\nabla H(\phi_i(x))| dx$$

Classification as minimization

Minimize

$$F(\phi_1, \ldots, \phi_K) = \alpha F^A(\phi_1, \ldots, \phi_K) + \beta F^B(\phi_1, \ldots, \phi_K) + \gamma F^C(\phi_1, \ldots, \phi_K)$$

with respect to all ϕ_i.

Experiment