Invariants to convolution

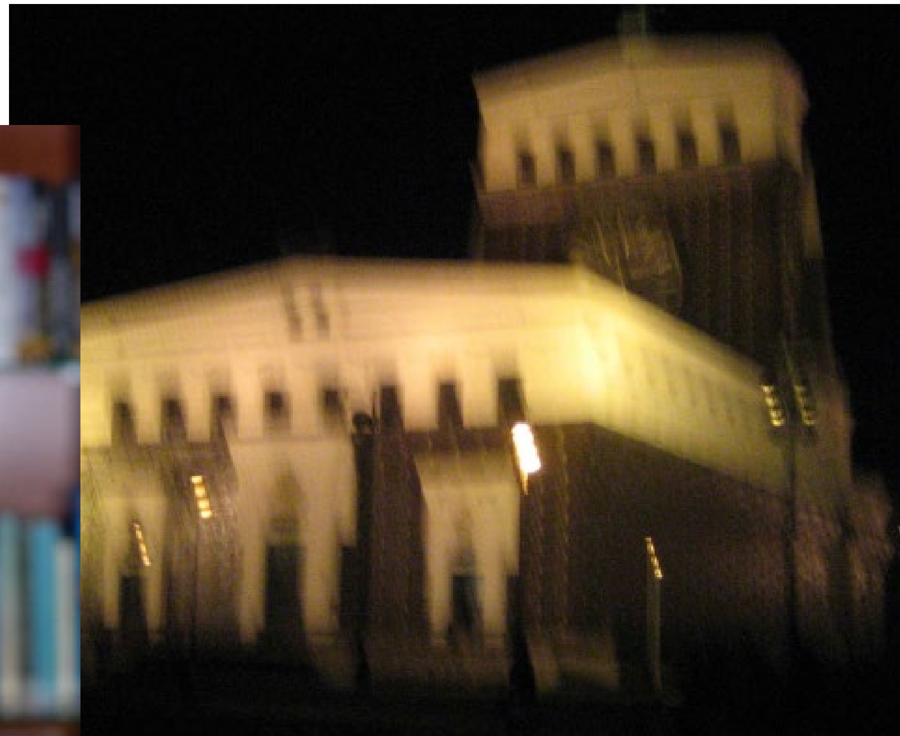
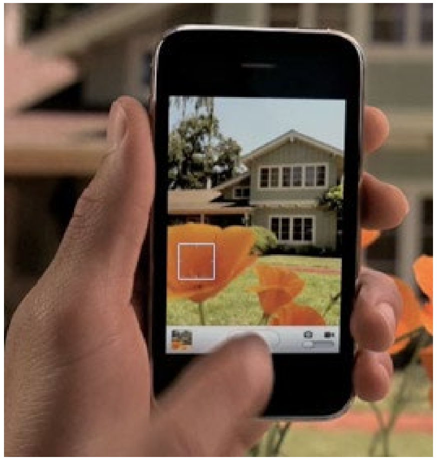
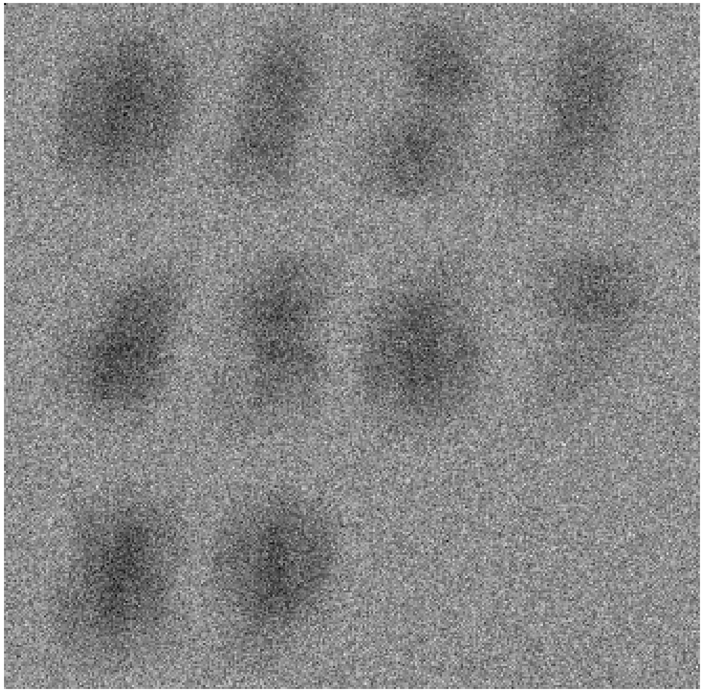


Image degradation by blurring



Motivation – character recognition



Motivation – template matching

(a)

Motivation – image registration

Simple blur model

- original *f(x)*

- acquired blurred image *g(x)*

$$g(x, y) = (f * h)(x, y)$$

h(x,y) is a PSF of the camera

Image degradation models

Space-variant

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)h(x,y,s,t)\ \mathrm{d}s\ \mathrm{d}t + n(x,y)$$

Space-invariant → convolution

$$h(x,y,s,t) = h(x-s, y-t)$$

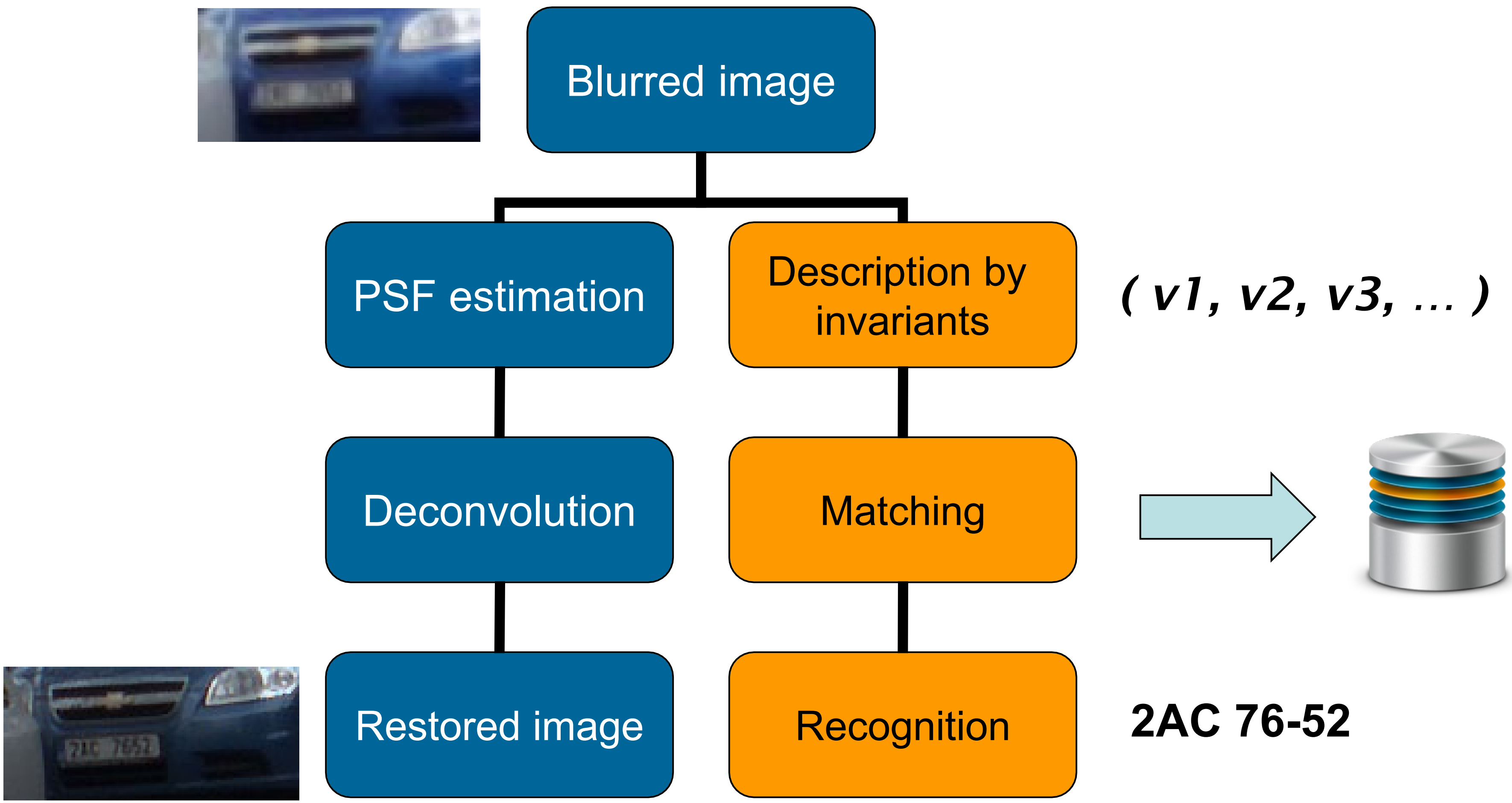
$$g(x,y) = (f * h)(x,y) + n(x,y)$$

Two approaches

Traditional approach: Image restoration (blind deconvolution) and traditional features

Proposed approach: Invariants to convolution

Direct analysis flowchart



Invariants to convolution

$$g(x,y) = (f * h)(x,y)$$

$$I(f * h) = I(f)$$

for any admissible *h*

The moments under convolution

Geometric/central

$$\mu_{pq}^{(g)} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} \mu_{kj}^{(h)} \mu_{p-k,q-j}^{(f)}$$

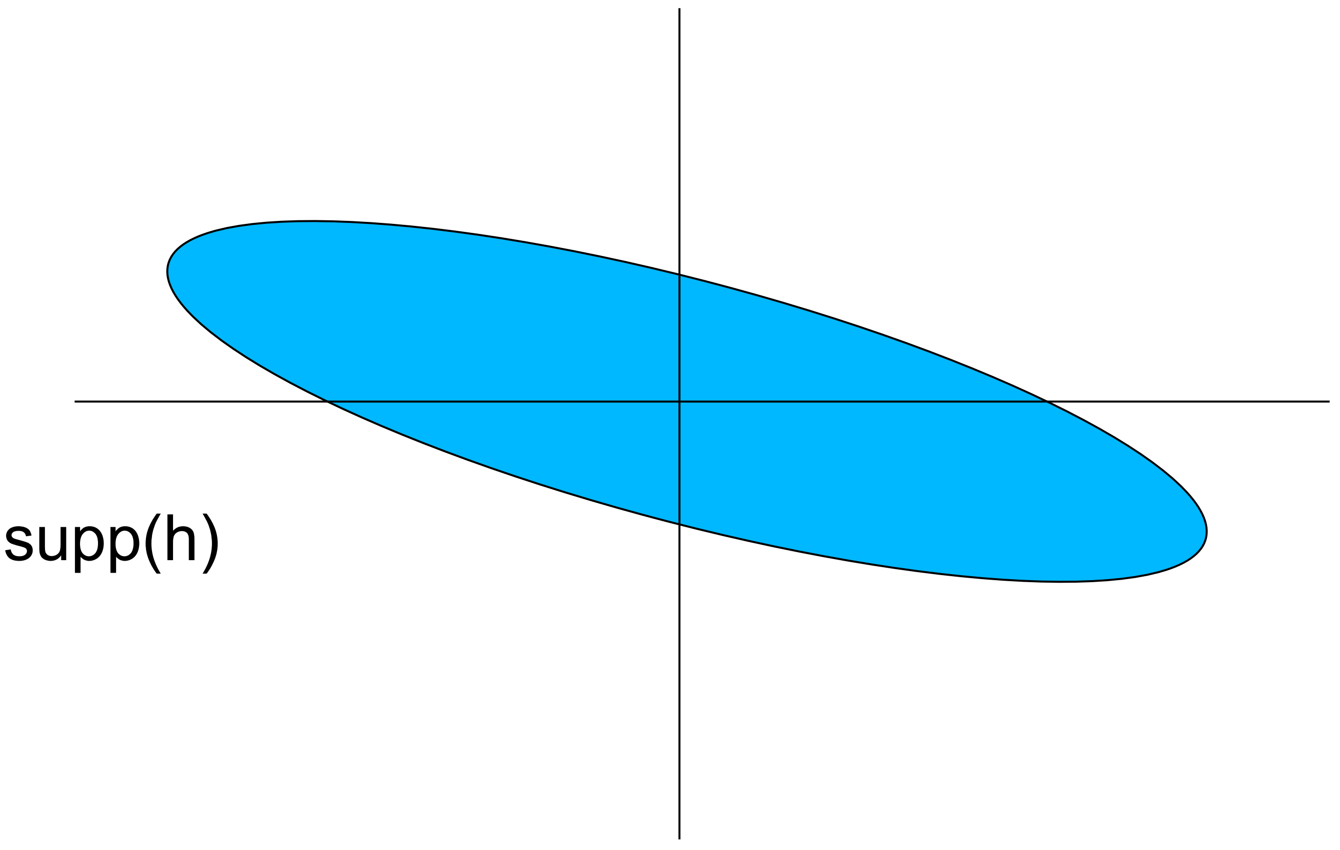
Complex

$$c_{pq}^{(f)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+iy)^p (x-iy)^q f(x,y) dx dy$$

$$c_{pq}^{(g)} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} c_{kj}^{(h)} c_{p-k,q-j}^{(f)}$$

Assumptions on the PSF

PSF is centrosymmetric ($N = 2$) and has a unit integral



Intuition: How to eliminate $c_{kj}^{(h)}$?

$$c_{pq}^{(g)} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} c_{kj}^{(h)} c_{p-k,q-j}^{(f)}$$

$$c_{00}^{(g)} = c_{00}^{(f)} c_{00}^{(h)} = c_{00}^{(f)}$$

$$c_{10}^{(g)} = c_{10}^{(f)} c_{00}^{(h)} + c_{00}^{(f)} c_{10}^{(h)} = c_{10}^{(f)} + c_{00}^{(f)} c_{10}^{(h)}$$

$$c_{20}^{(g)} = c_{20}^{(f)} + 2c_{10}^{(f)} c_{10}^{(h)} + c_{00}^{(f)} c_{20}^{(h)}$$

$$c_{30}^{(g)} = c_{30}^{(f)} + 3c_{20}^{(f)} c_{10}^{(h)} + 3c_{10}^{(f)} c_{20}^{(h)} + c_{00}^{(f)} c_{30}^{(h)}$$

Invariants to centrosymmetric convolution

$$
\begin{aligned}
C(3,0) &= \mu_{30}, \\
C(2,1) &= \mu_{21}, \\
C(1,2) &= \mu_{12}, \\
C(0,3) &= \mu_{03}.
\end{aligned}
$$

Invariants to centrosymmetric convolution

$$
\begin{aligned}
C(5,0) &= \mu_{50} - \frac{10\mu_{30}\mu_{20}}{\mu_{00}}, \\
C(4,1) &= \mu_{41} - \frac{2}{\mu_{00}}(3\mu_{21}\mu_{20} + 2\mu_{30}\mu_{11}), \\
C(3,2) &= \mu_{32} - \frac{1}{\mu_{00}}(3\mu_{12}\mu_{20} + \mu_{30}\mu_{02} + 6\mu_{21}\mu_{11}), \\
C(2,3) &= \mu_{23} - \frac{1}{\mu_{00}}(3\mu_{21}\mu_{02} + \mu_{03}\mu_{20} + 6\mu_{12}\mu_{11}), \\
C(1,4) &= \mu_{14} - \frac{2}{\mu_{00}}(3\mu_{12}\mu_{02} + 2\mu_{03}\mu_{11}), \\
C(0,5) &= \mu_{05} - \frac{10\mu_{03}\mu_{02}}{\mu_{00}}.
\end{aligned}
$$

Invariants to centrosymmetric convolution

$$C(p,q)^{(f)} = \mu_{pq}^{(f)} - \frac{1}{\mu_{00}^{(f)}} \sum_{\substack{n=0 \\ 0<n+m<p+q}}^{p} \sum_{m=0}^{q} \binom{p}{n}\binom{q}{m} C(p-n,q-m)^{(f)} \cdot \mu_{nm}^{(f)}$$

$$K(p,q)^{(f)} = c_{pq}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} K(p-n,q-m)^{(f)} \cdot c_{nm}^{(f)}$$

where $(p + q)$ is odd

What is the intuitive meaning of the invariants?

“Measure of anti-symmetry”

Convolution invariants in FT domain

$$\begin{aligned} g &= f * h \\ G &= F \cdot H \\ |G| &= |F| \cdot |H| \\ \mathrm{ph} G &= \mathrm{ph} F + \mathrm{ph} H \end{aligned}$$

Convolution invariants in FT domain

$$\text{Centrosymmetric } h(x,y) \Longrightarrow \text{real } H(u,v)$$

$$\mathrm{ph}H \in \{0;\pi\}$$

$$\tan(\mathrm{ph}G) = \tan(\mathrm{ph}F)$$

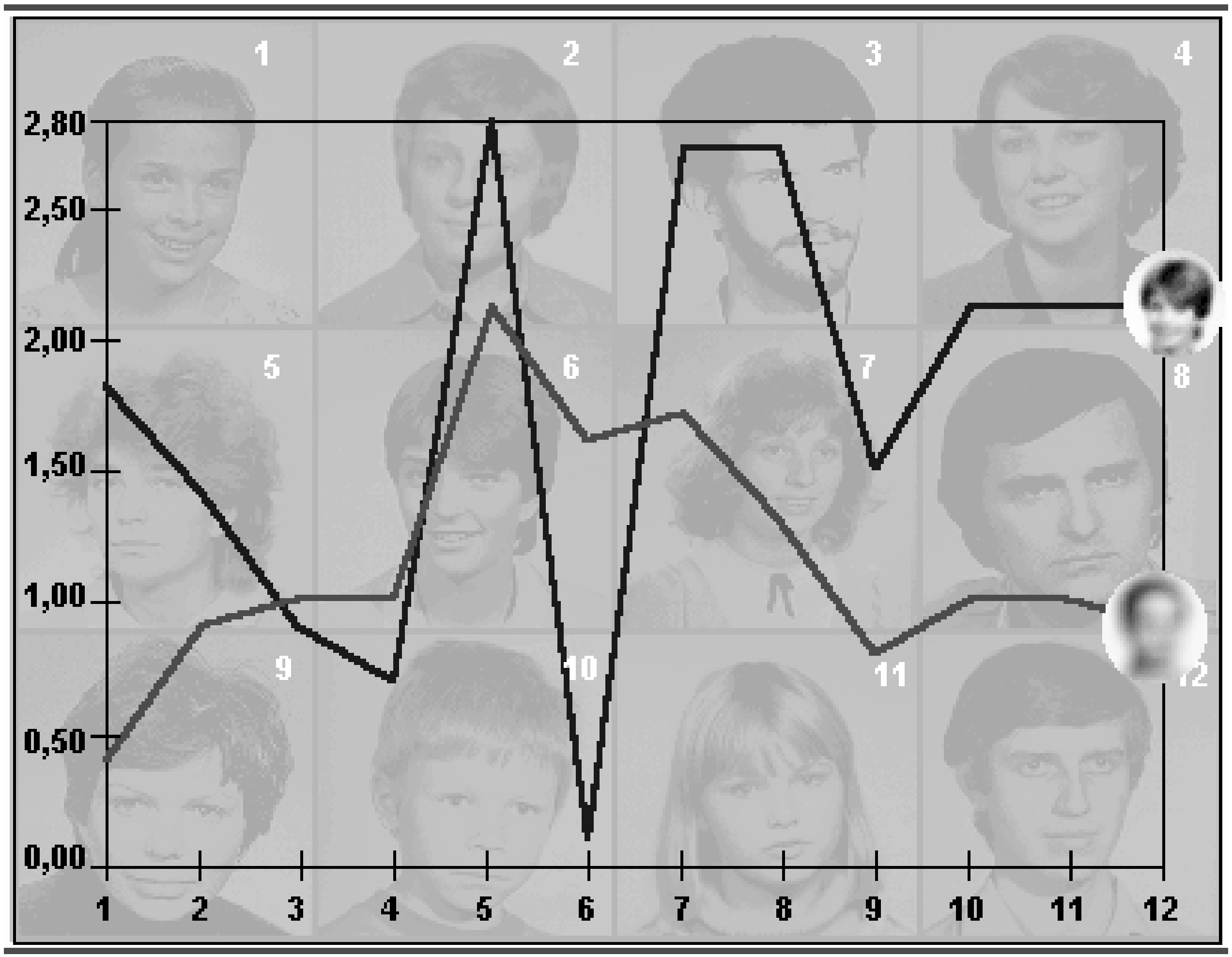
Relationship between FT and moment invariants

$$\tan(\mathrm{ph}F(u,v)) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{kj} u^k v^j,$$

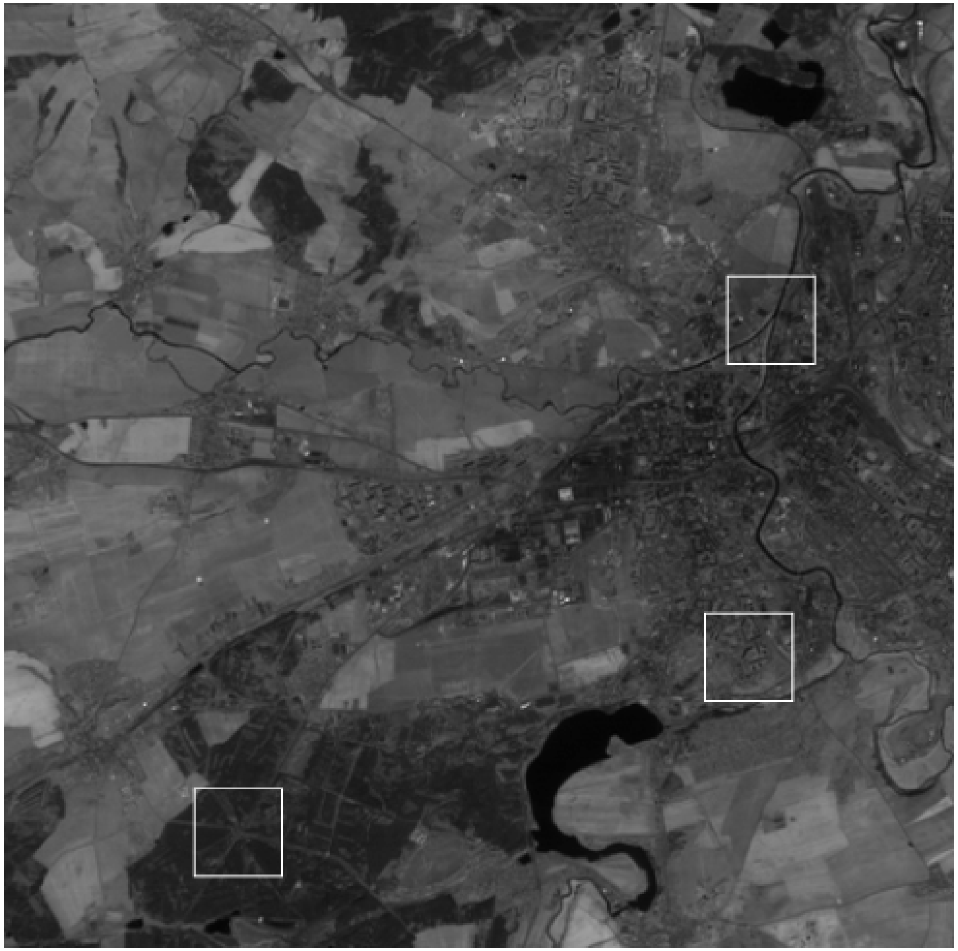
$$a_{kj} = \frac{(-1)^{(k+j-1)/2} \cdot (-2\pi)^{k+j}}{k! \cdot j! \cdot \mu_{00}} M(k,j)^{(f)}.$$

where *M(k,j)* is the same as *C(k,j)* but with geometric moments

Face recognition – simulated example



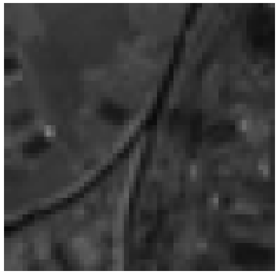
Template matching

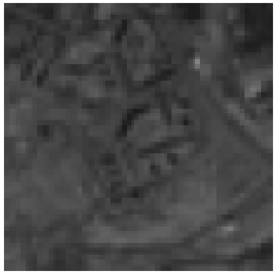


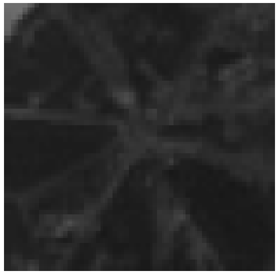
sharp image

with the templates

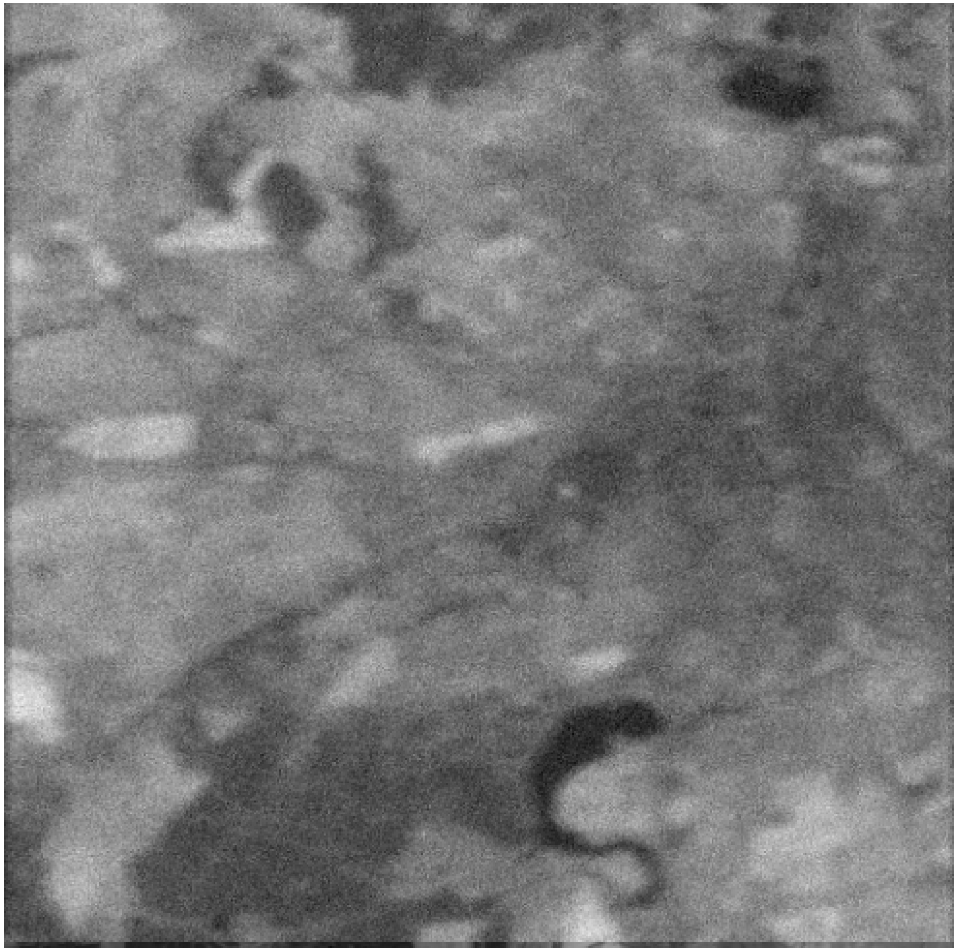
the templates (close-up)





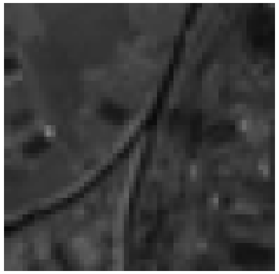


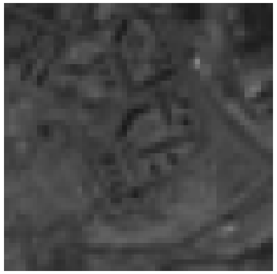
Template matching

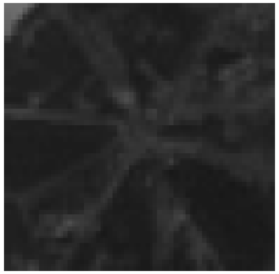


a frame where the templates were located successfully

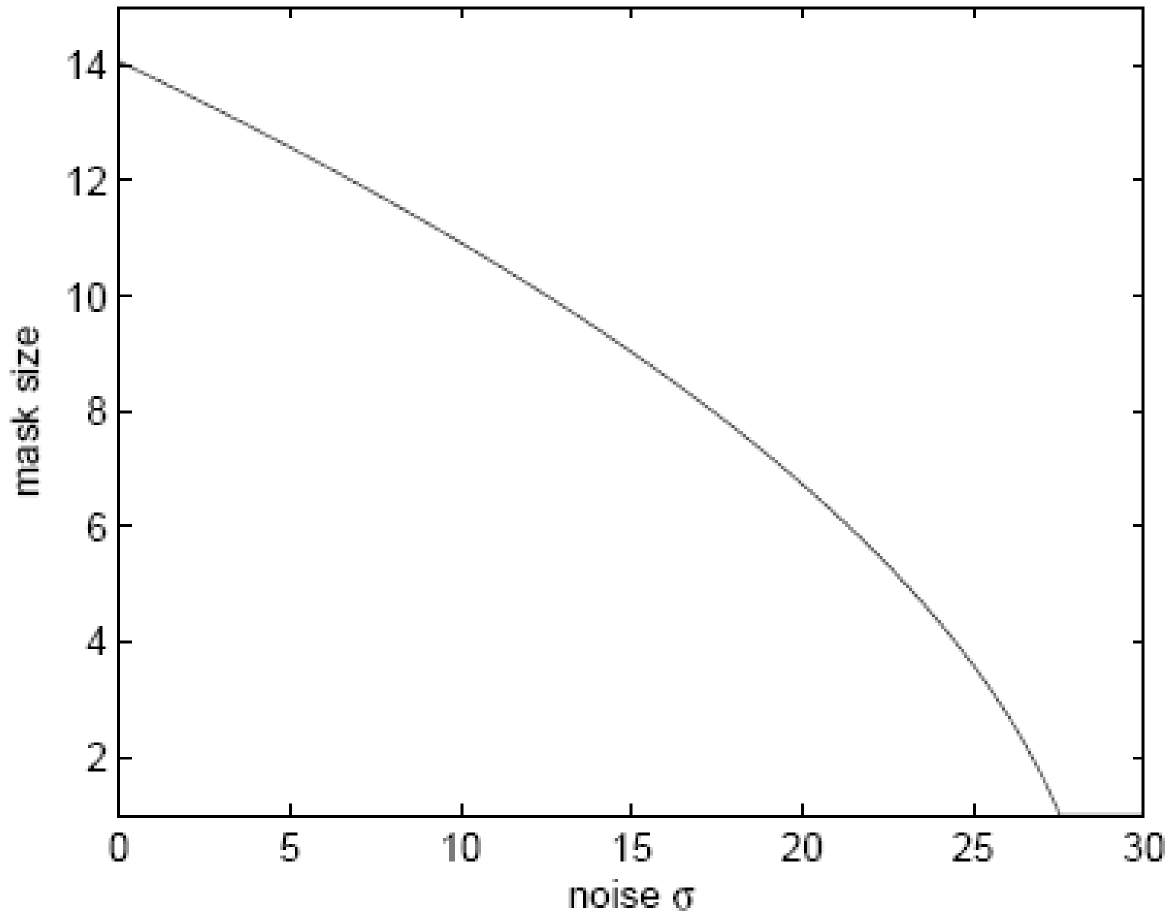
the templates (close-up)







Template matching performance



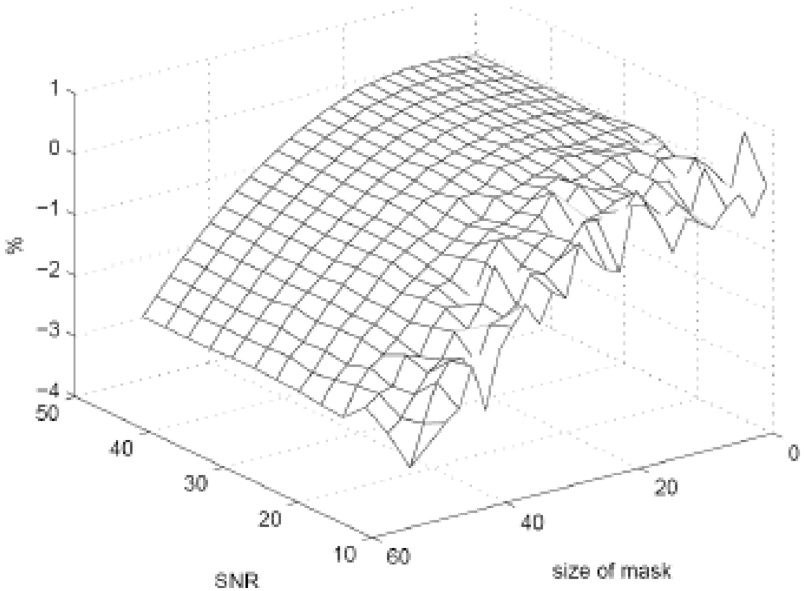
Boundary effect in template matching

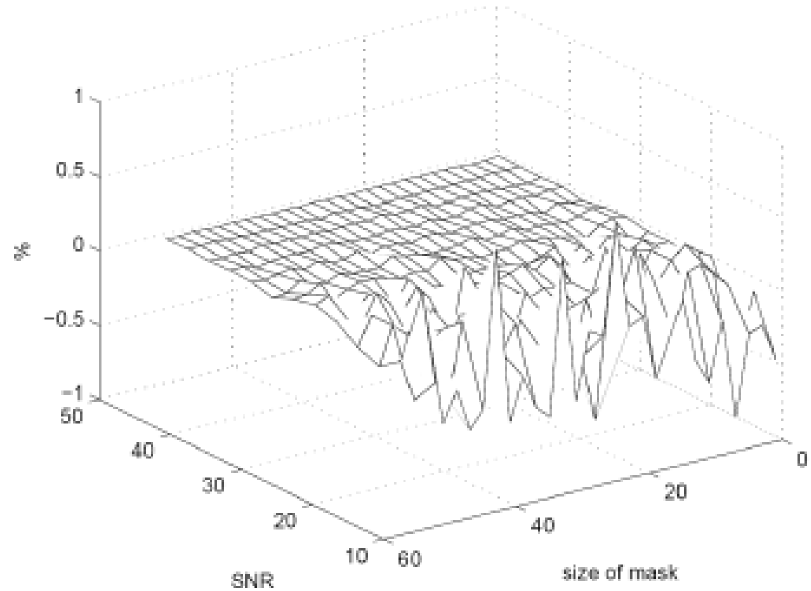
Valid region

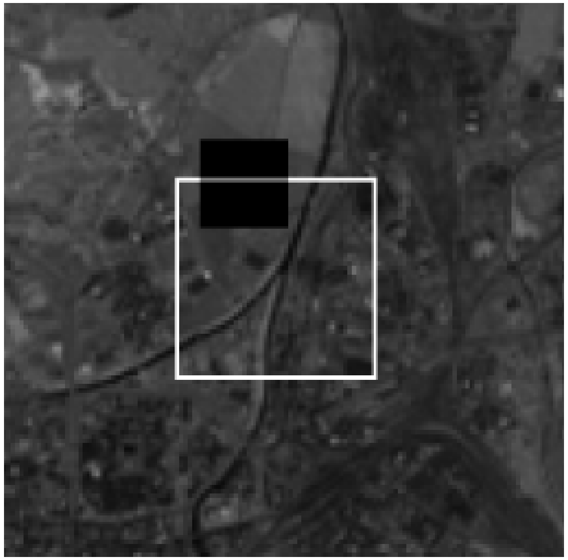
Boundary effect in template matching

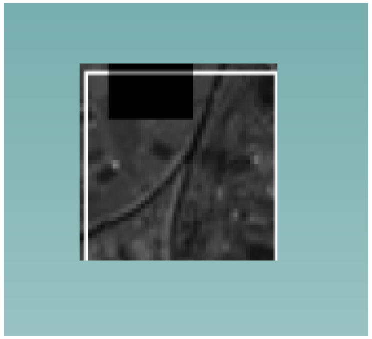
Invalid region
- invariance is violated

Boundary effect in template matching



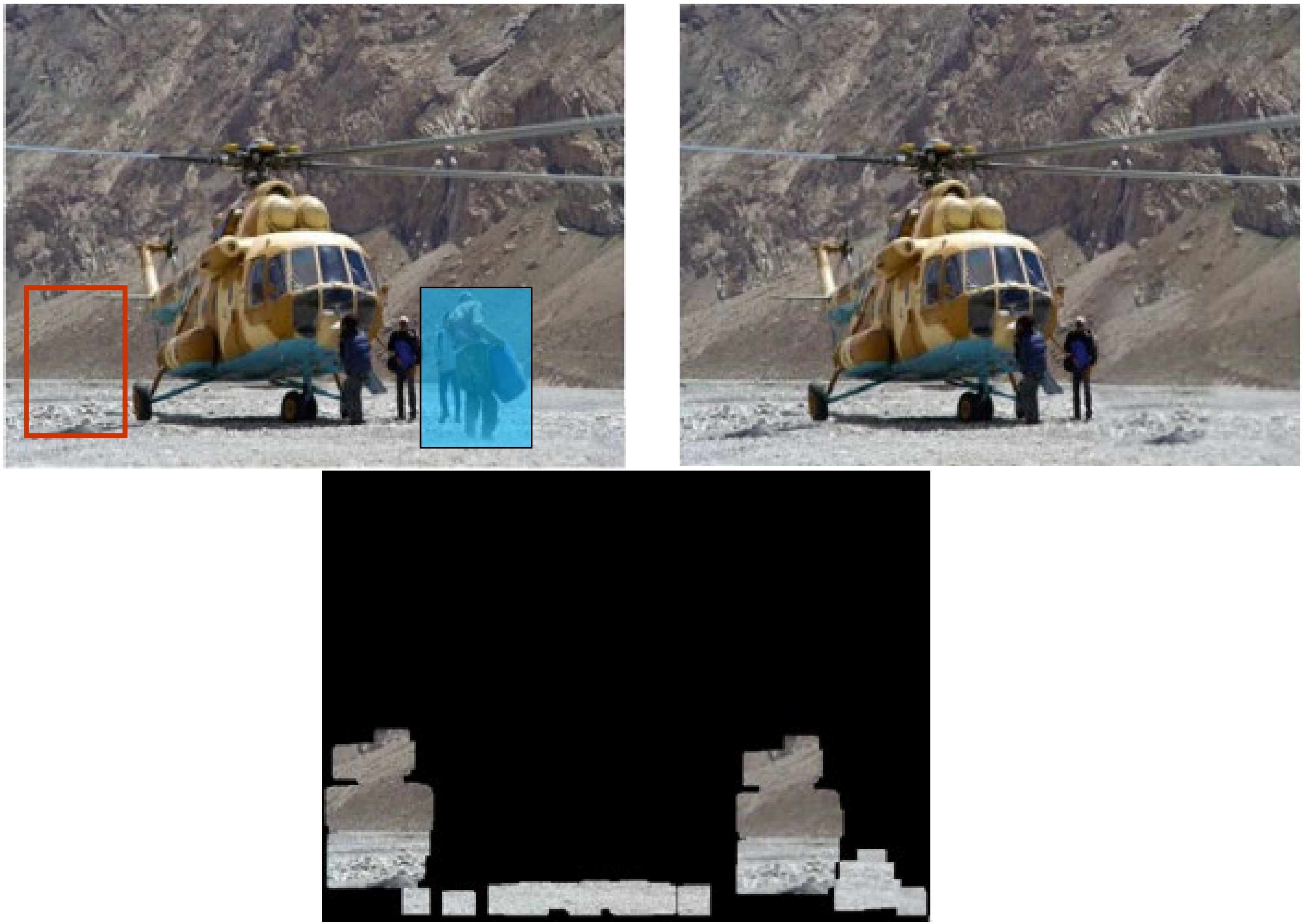






zero-padding

Application in detecting forgeries



Other types of the blurring PSF

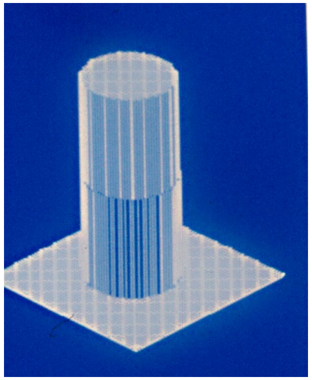
- *N*-fold rotation symmetry, $N > 2$
- Axial symmetry
- Circular symmetry
- Gaussian PSF
- Motion blur PSF

The more we know about the PSF, the more invariants and the higher discriminability we get

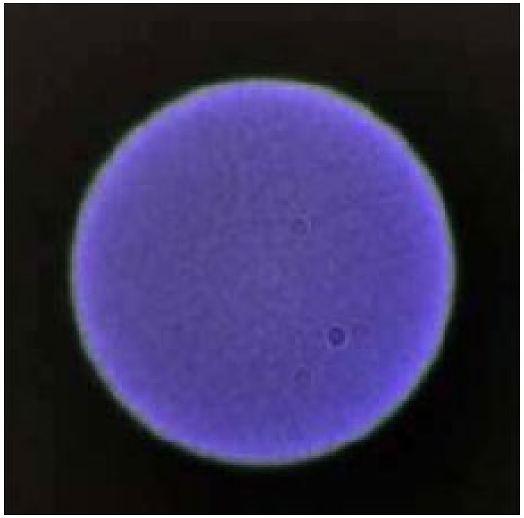
Assumptions on the PSF

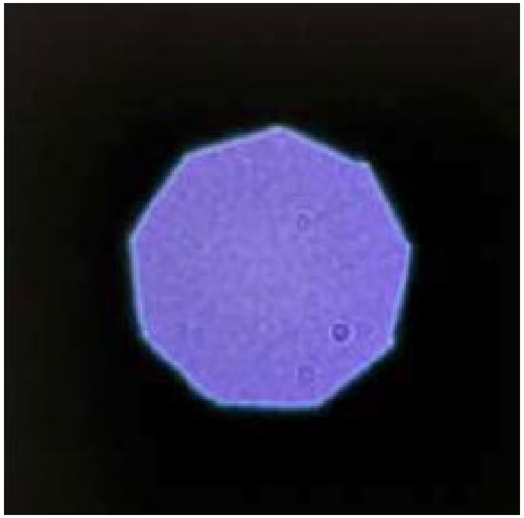
Out-of-focus blur, "geometric optic approximation"

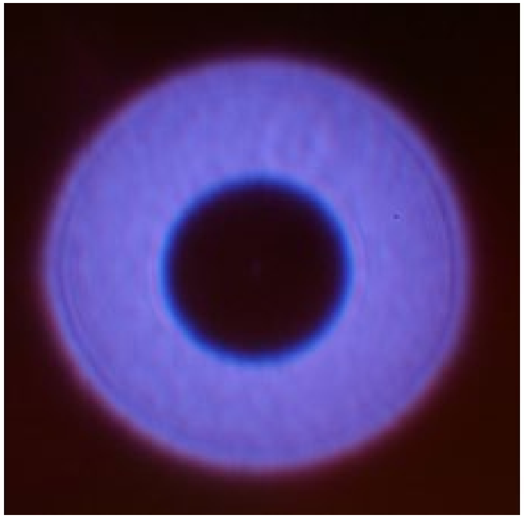
$$g(x, y) = (f * h)(x, y)$$

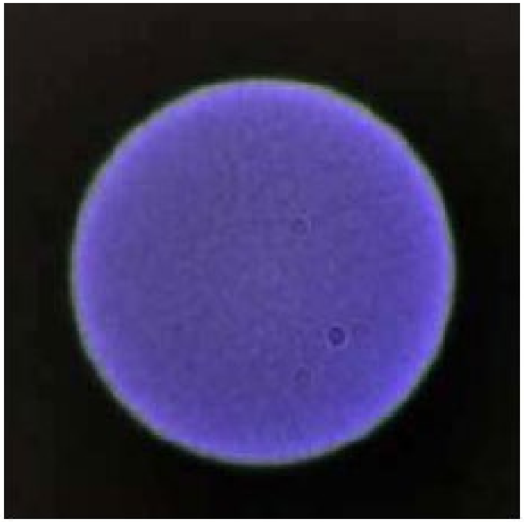
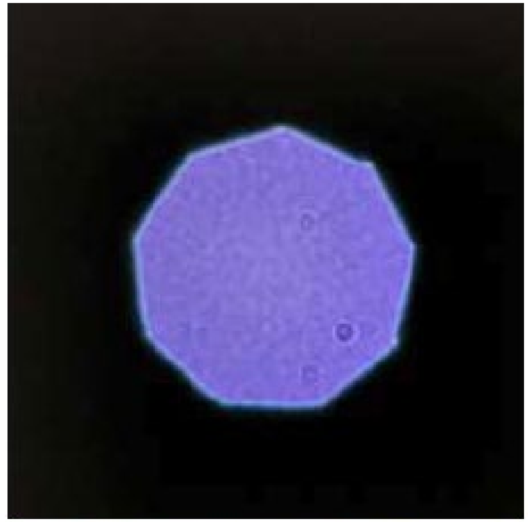
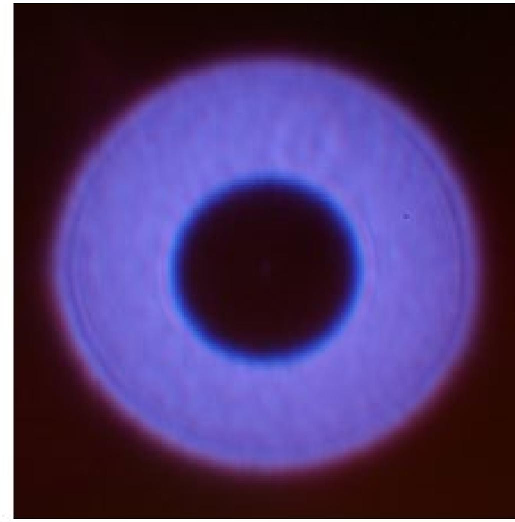


Is it realistic?



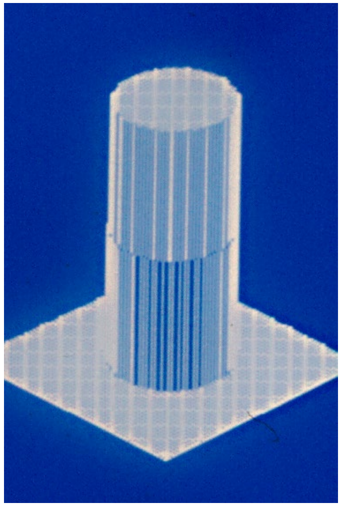


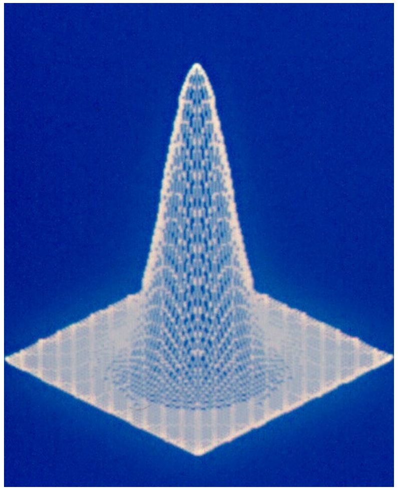



(Theoretical diaphragm - 8 blades case)

Invariants to circularly symmetric PSF

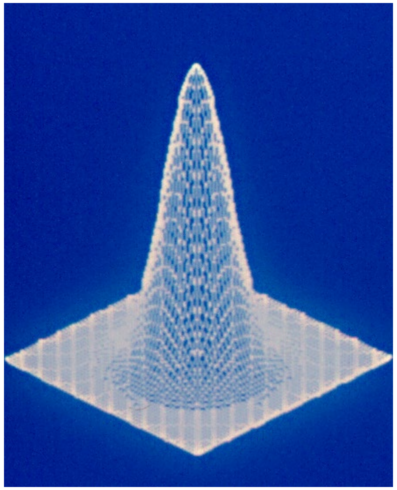




$$K_\infty(p,q)^{(f)} = c_{pq}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{n=1}^{q} \binom{p}{n} \binom{q}{n} K_\infty(p-n, q-n)^{(f)} \cdot c_{nn}^{(f)}$$

If $p = q$ → the invariant is zero (except $p = q = 0$)

Invariants to Gaussian PSF



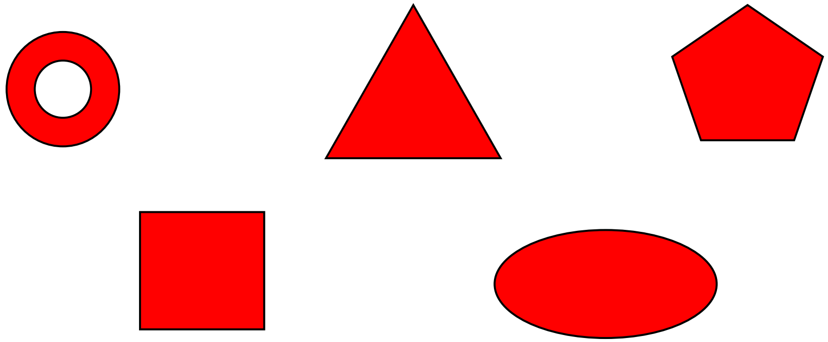
$$c_{pq}^{(G_\sigma)} = \begin{cases} (2\sigma^2)^p p! & p = q \\ 0 & p \neq q \end{cases}$$

$$K_G(p,q)^{(f)} = c_{pq}^{(f)} - \sum_{k=1}^{q} k! \binom{p}{k} \binom{q}{k} \left(\frac{c_{11}^{(f)}}{c_{00}^{(f)}} \right)^k K_G(p-k, q-k)^{(f)}$$

$$K_G(p,q)^{(f)} = \sum_{j=0}^{q} j! \binom{p}{j} \binom{q}{j} \left(-\frac{c_{11}^{(f)}}{c_{00}^{(f)}} \right)^j c_{p-j,q-j}^{(f)}$$

If $p = q = 1$ → the invariant is zero

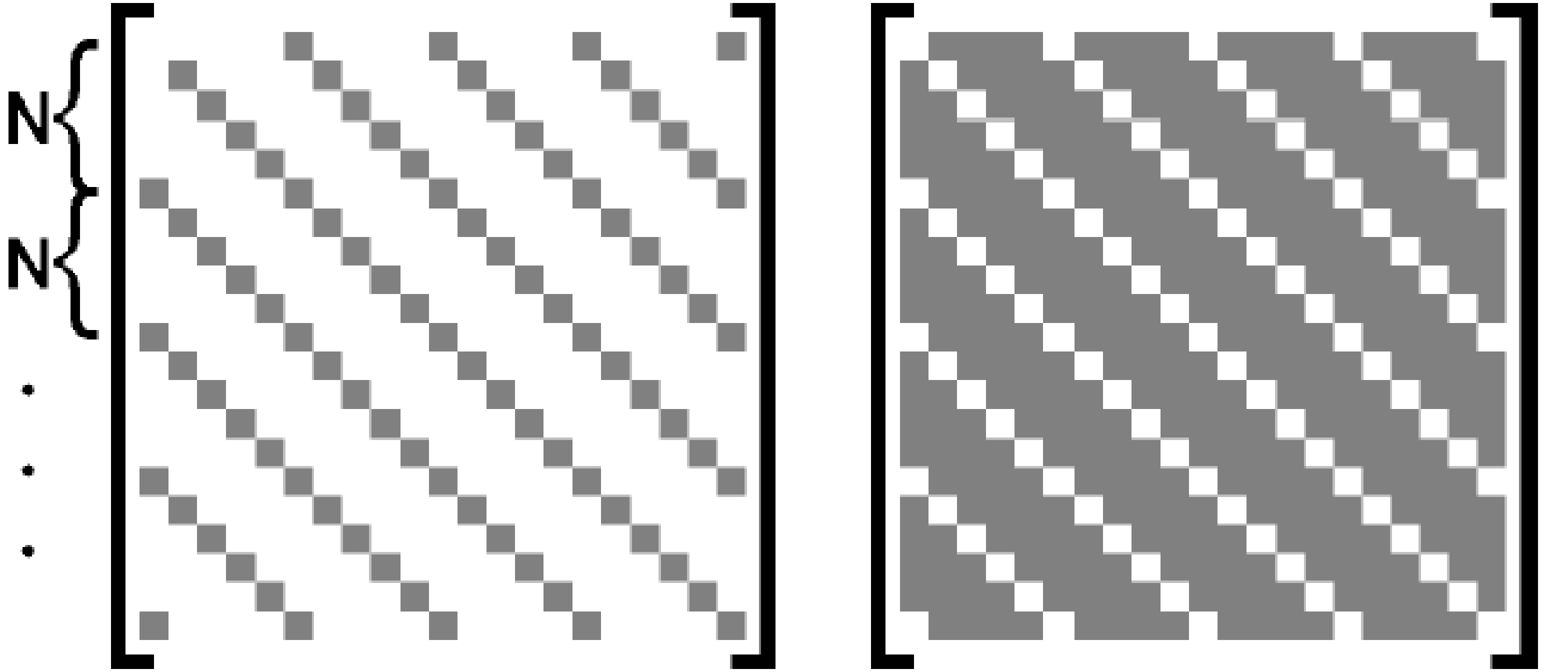
Invariants for arbitrary N



$$K_N(p,q) =$$

$$= c_{pq} - \frac{1}{c_{00}} \sum_{j=0}^{p} \sum_{\substack{k=0 \\ 0<j+k \\ (j-k)/N \text{ is integer}}}^{q} \binom{p}{j} \binom{q}{k} K_N(p-j, q-k) \cdot c_{jk}.$$

Invariants for arbitrary *N*



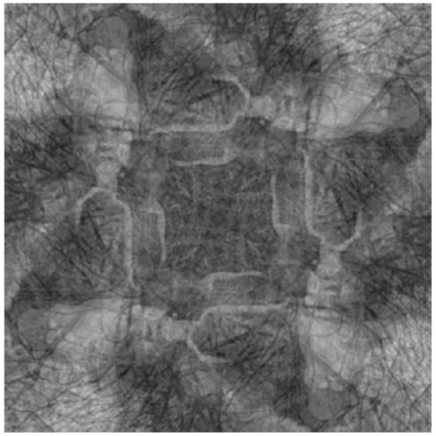
If $(p-q)/N$ is integer → the invariant is zero

Projection operators

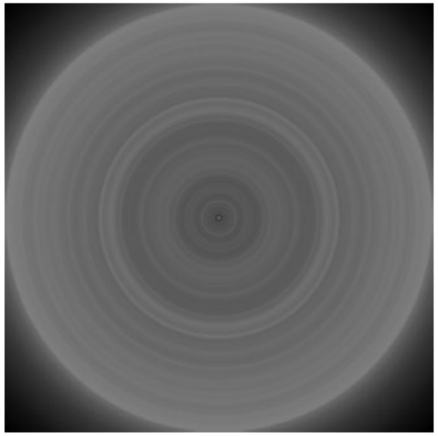
$$(P_N f)(r,\theta) = \frac{1}{N}\sum_{j=1}^{N} f(r, \theta + \alpha_j)$$

(a)

(b)



(c)



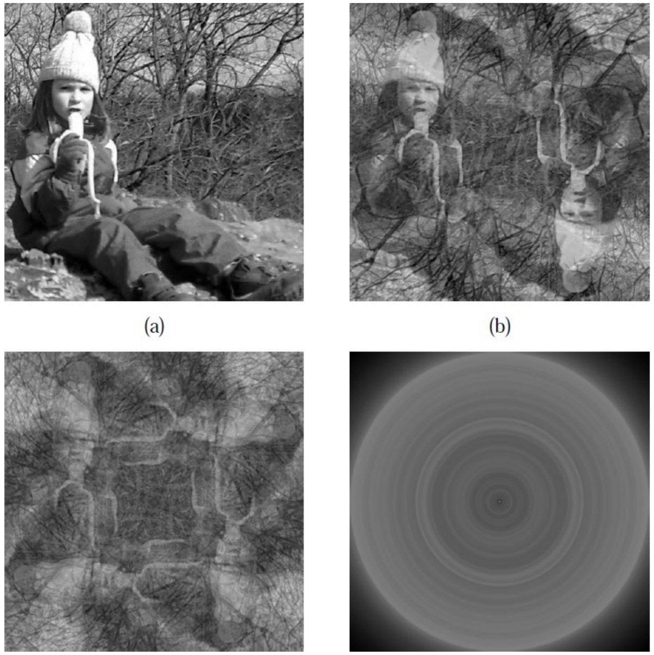
(d)

Blur invariants in Fourier domain

$$G = F \cdot H$$

$$I_N(u,v) = \frac{\mathcal{F}(f)(u,v)}{\mathcal{F}(P_N f)(u,v)}$$

$$I_N^{(f)} = I_N^{(f*h)}$$



Blur invariants by image moments

$$I_N(U,V) = \frac{F}{P_N F}(U,V) = \sum_{j,k=0}^{\infty} \frac{(-2\pi i)^{j+k}}{j!k!} A_{jk} u^j v^k$$

$$A_{pq} = \frac{c_{pq}^{(f)}}{c_{00}^{(f)}} - \sum_{\substack{j=0 \\ 0<j+k \\ (j-k)/N \text{ is integer}}}^{p} \sum_{k=0}^{q} \binom{p}{j} \binom{q}{k} \frac{c_{jk}^{(f)}}{c_{00}^{(f)}} A_{p-j,q-k}$$

Discrimination power

The null-space of the blur invariants = a set of all images having the same symmetry as the PSF.

One cannot distinguish among symmetric objects.

Some other objects also cannot be distinguished.

Completeness theorem

If

$$K_N(p,q)^f = K_N(p,q)^g \qquad p,q = 0,1,\cdots$$

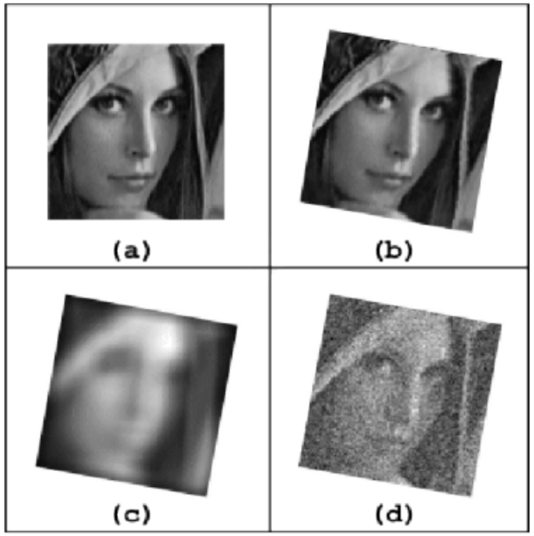
Then

$$f * h_1 = g * h_2$$

Recommendation for practice

It is important to learn as much as possible about the PSF and to use proper invariants for object recognition

Combined invariants



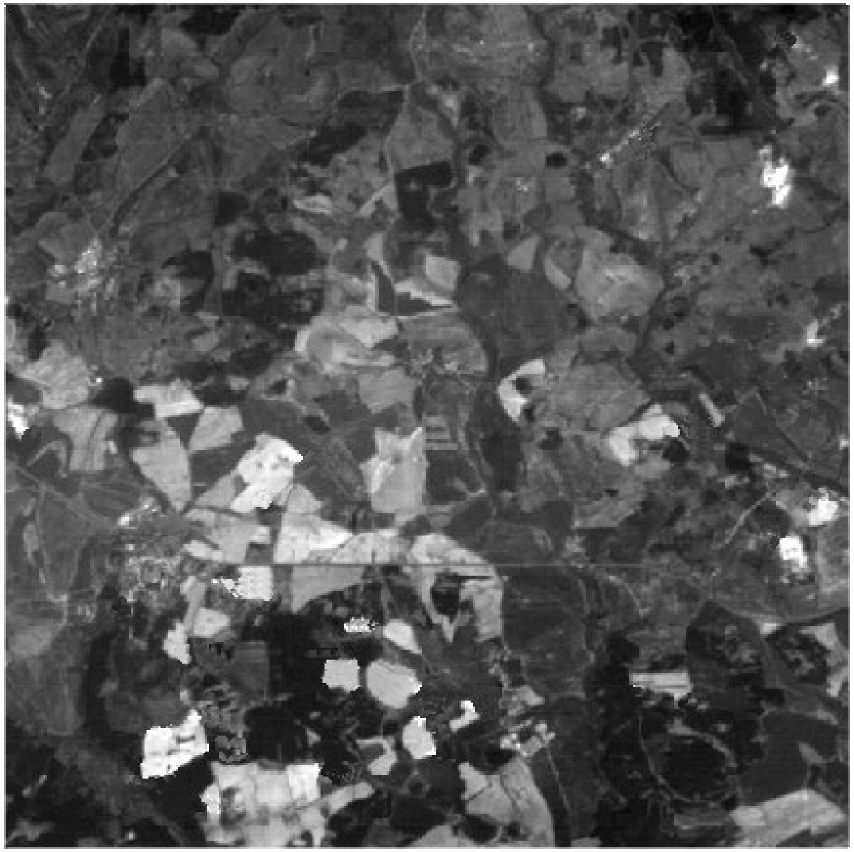
Convolution and rotation

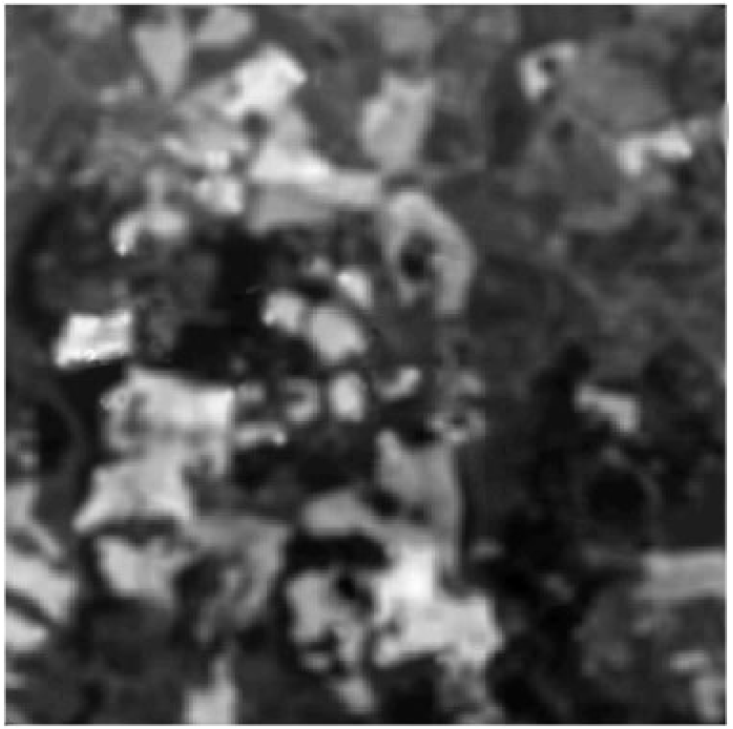
For any *N*

$$K'(p,q) = e^{-i(p-q)\alpha} \cdot K(p,q)$$

$$I = \prod_{j=1}^{n} K(p_j, q_j)^{k_j} \qquad \sum_{j=1}^{n} k_j (p_j - q_j) = 0$$

Satellite image registration by combined invariants

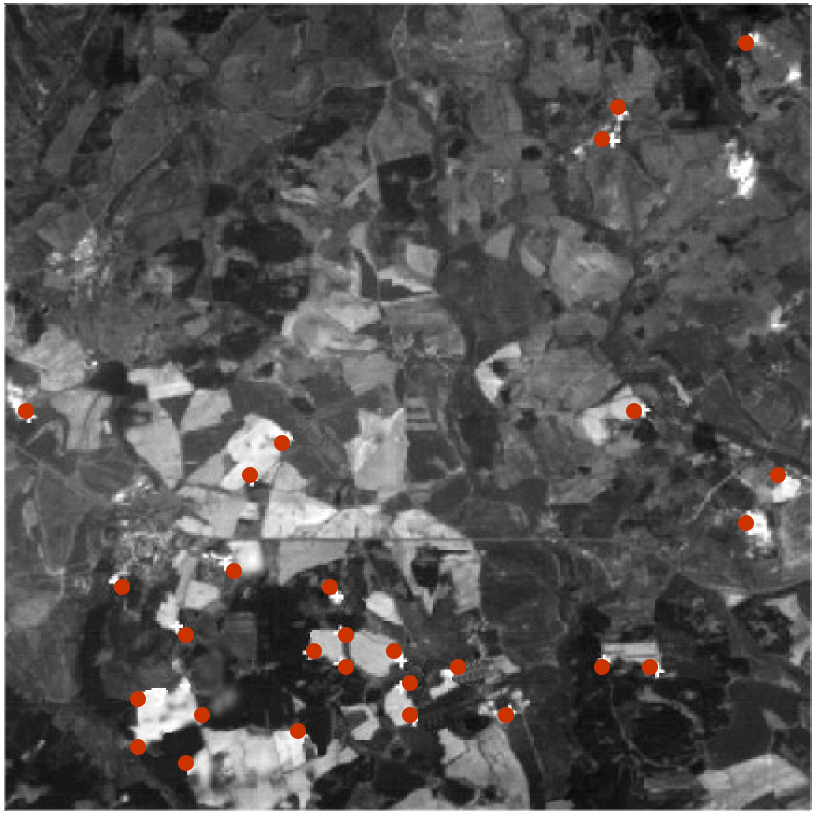


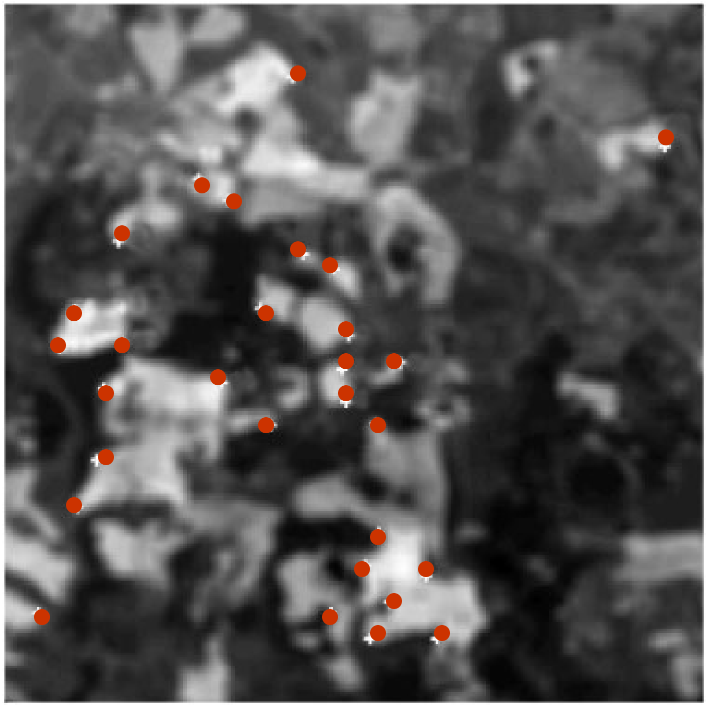


Registration algorithm

- Independent corner detection in both images
- Extraction of salient corner points
- Calculating blur-rotation invariants of a circular neighborhood of each extracted corner
- Matching corners by means of the invariants
- Refining the positions of the matched control points
- Estimating the affine transform parameters by a least-square fit
- Resampling and transformation of the sensed image

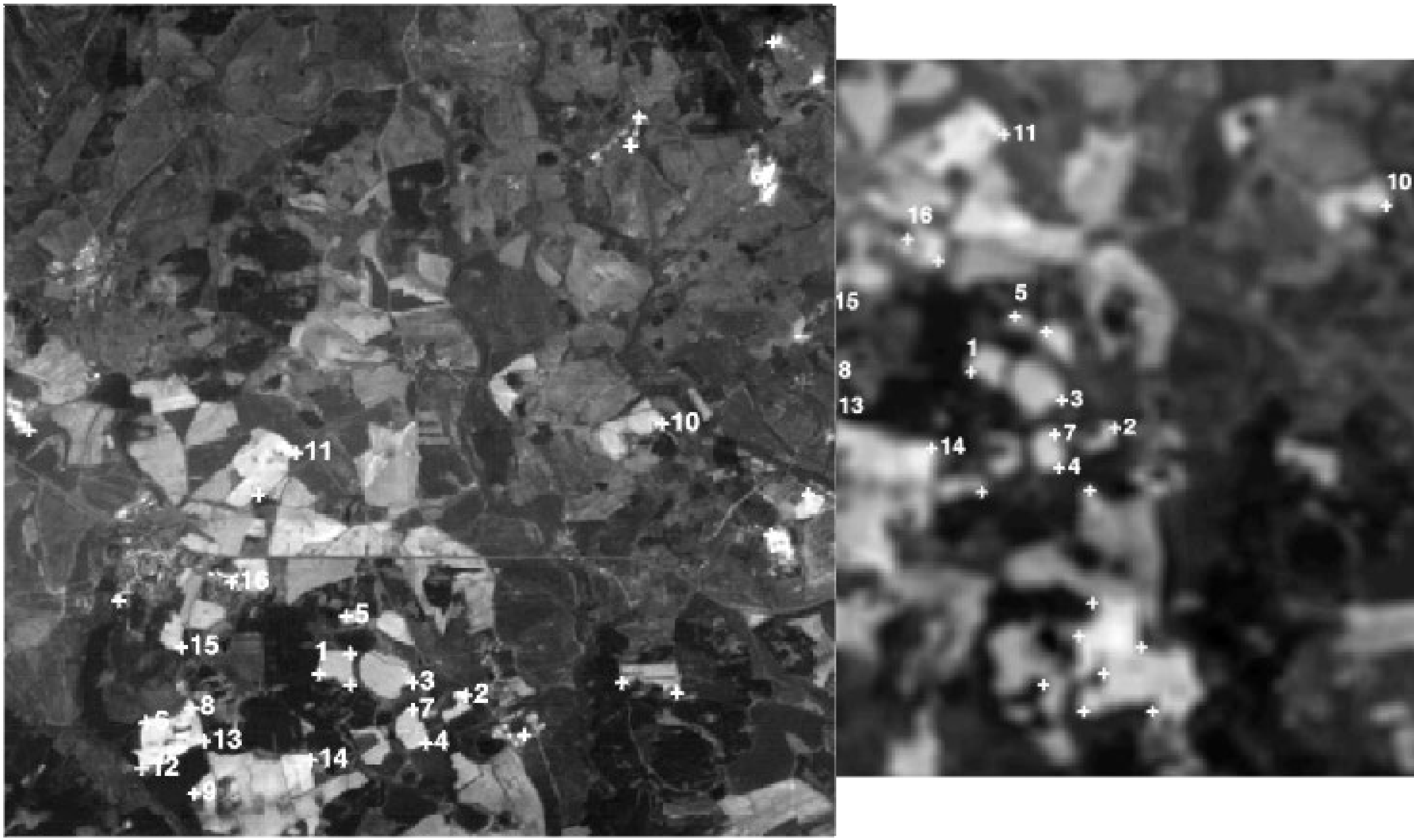
Control point detection



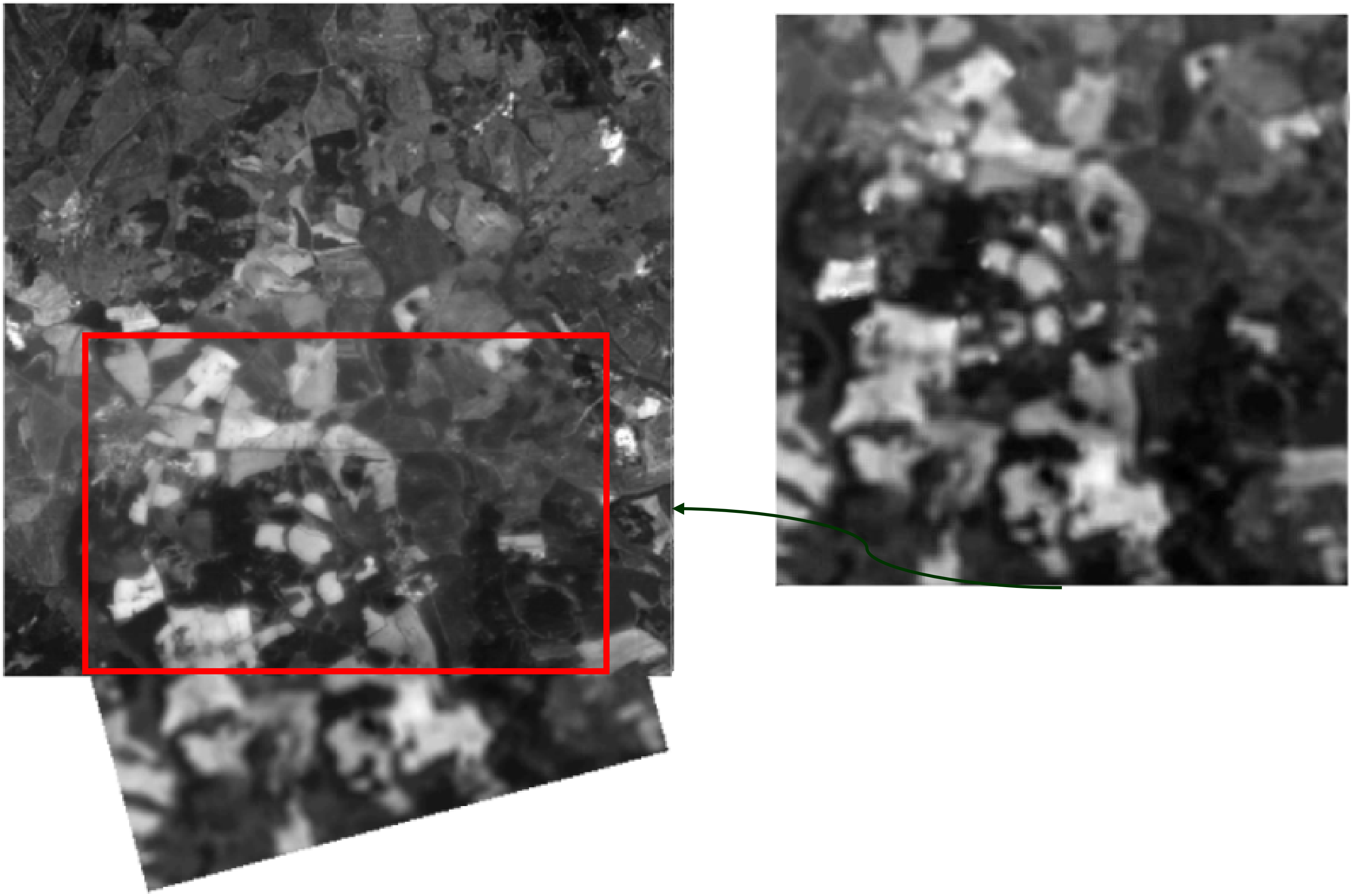


$$\min_{k,m} \text{distance}((\, v1_k,\, v2_k,\, v3_k,\, \ldots \,) ,\, (\, v1_m,\, v2_m,\, v3_m,\, \ldots \,))$$

Control point matching



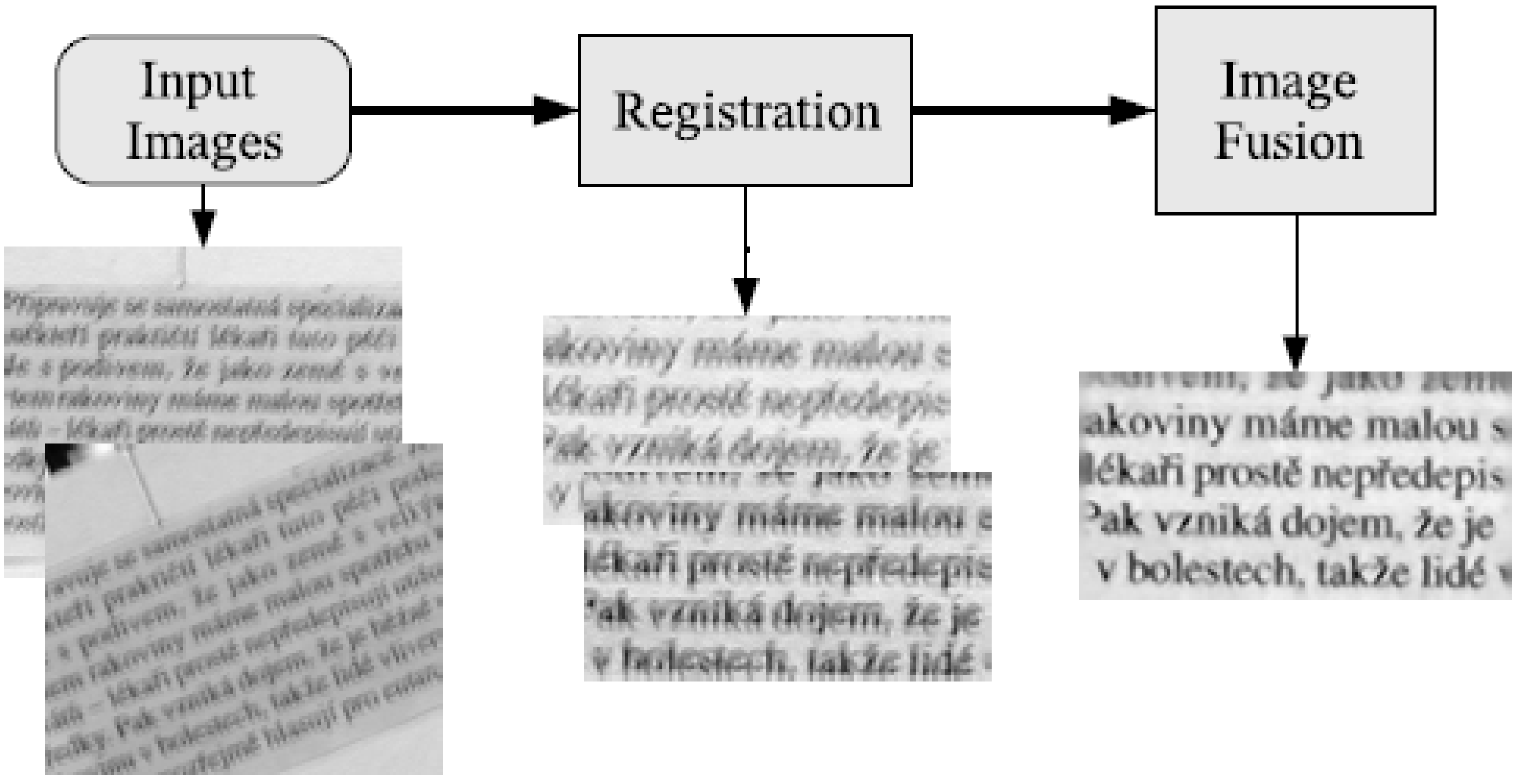
Registration result



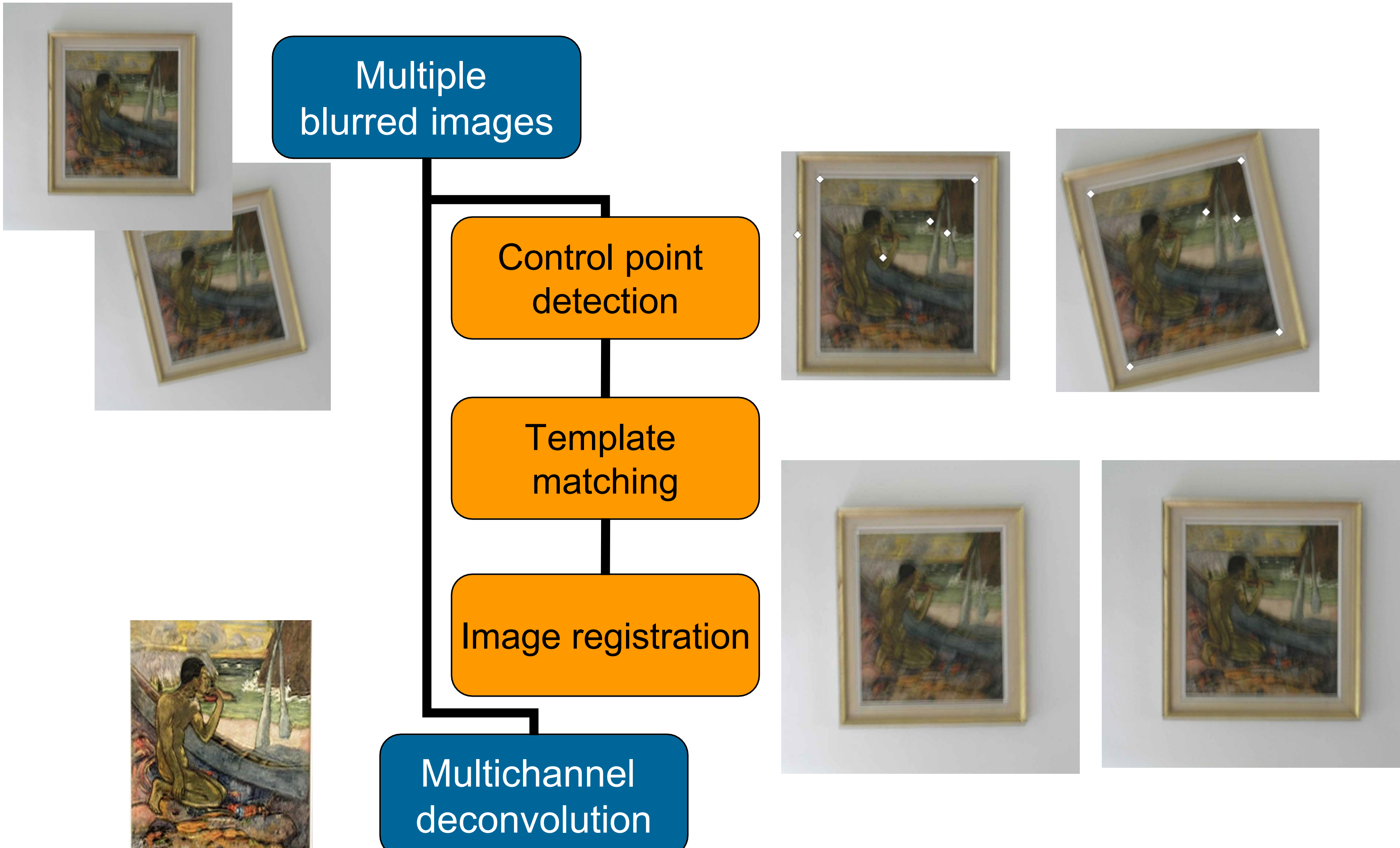
Camera motion estimation

Camera motion estimation

Application in MBD I



Invariants and deconvolution in a single task



Application in MBD II

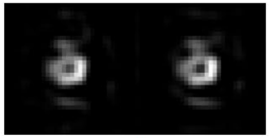
The Poor Fisherman, Paul Gauguin, 1896


1
2
2
1
2
1
2
1

(a)

(b)

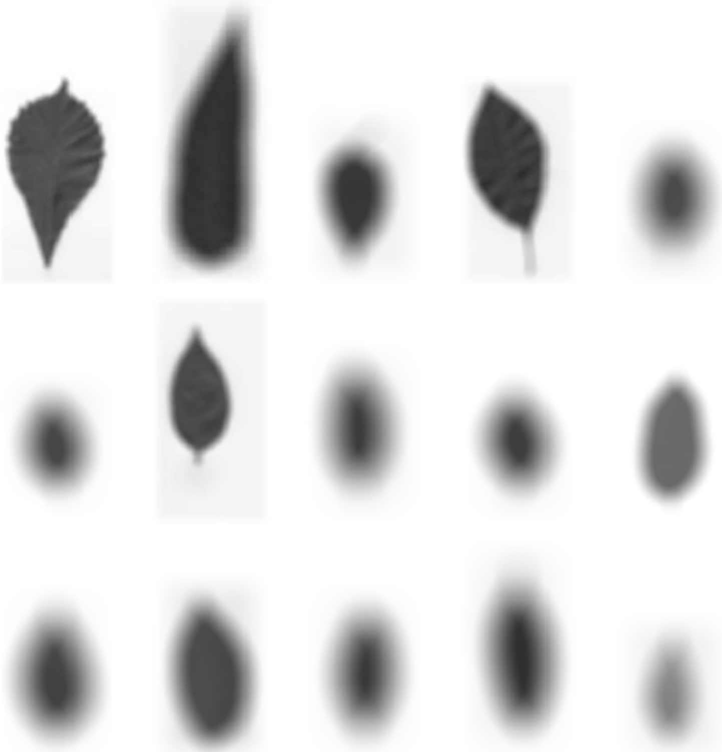
(c)



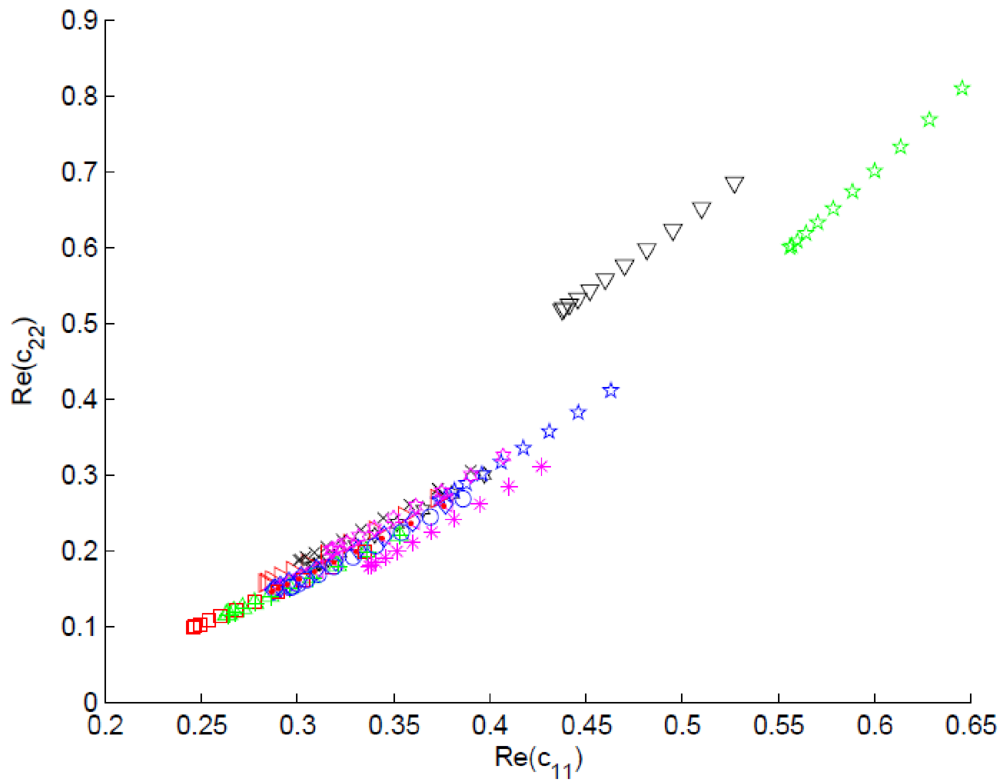
(d)

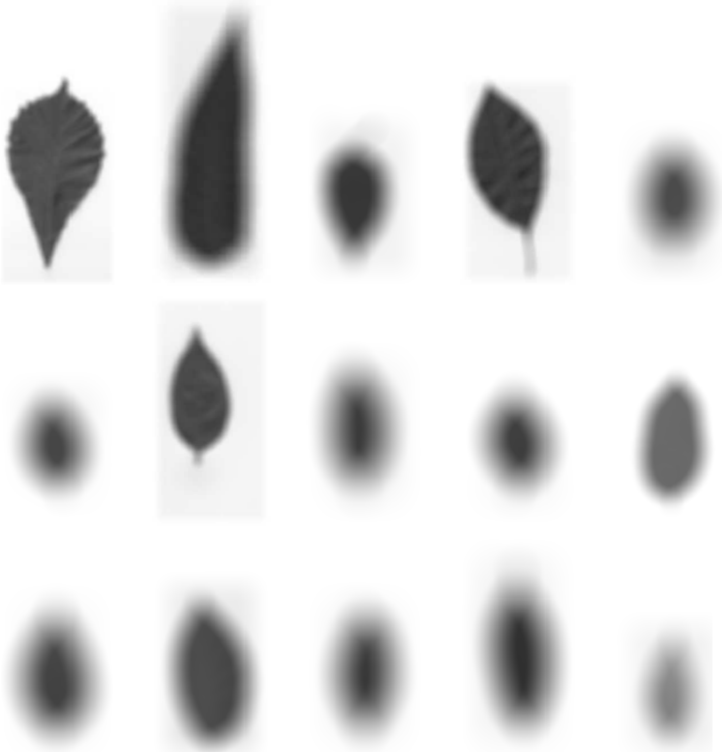
Leaf recognition system MEW2010

10 000 tree leaves
100 classes (species)
Recognition based solely on the contour

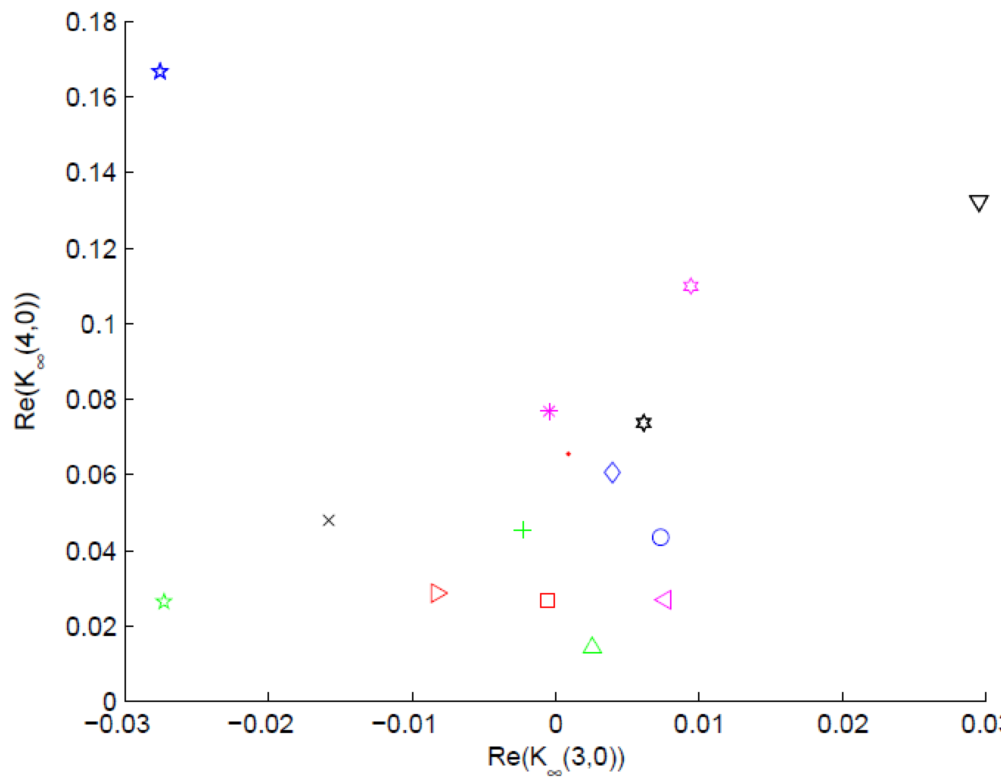


Plain moments





Invariants

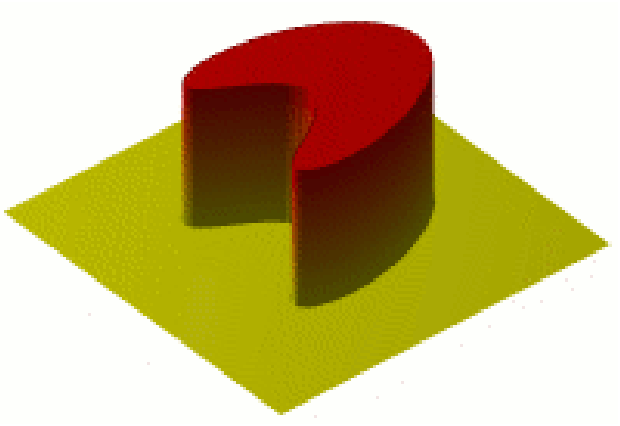

0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
−0.03
−0.02
−0.01
0
0.01
0.02
0.03
$Re(K_\infty(4,0))$
$Re(K_\infty(3,0))$

Documented applications of convolution and combined invariants

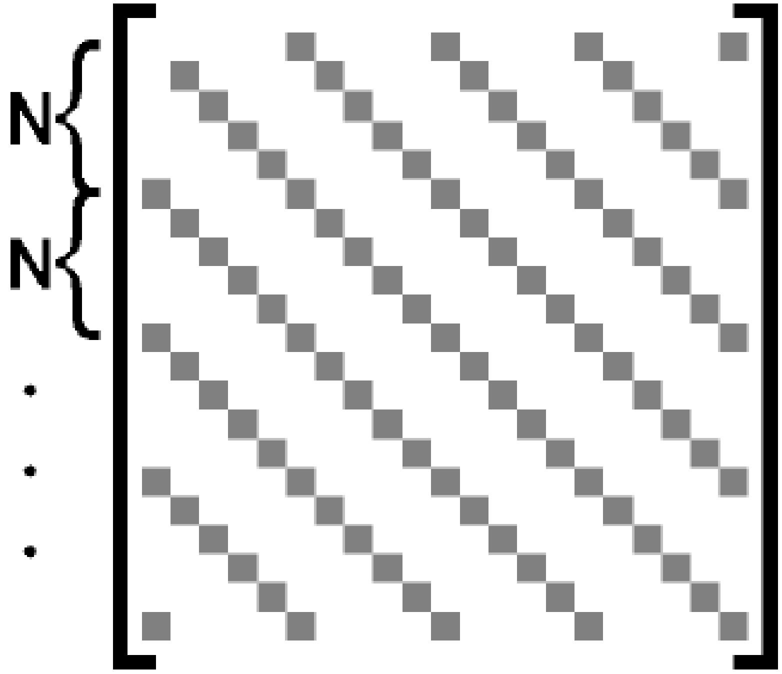
- Character/digit/symbol recognition in the presence of vibration, linear motion or out-of-focus blur
- Robust image registration (medical, satellite, ...)
- Detection of image forgeries

Blur invariants and heat equation

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$



"Dihedral" Invariants



If *(p-q)/N* is a non-zero integer → the invariant is non-trivial but constrained

"Dihedral" Invariants

$$D_{pq}^{N} = \frac{c_{pq}}{c_{00}} - \sum_{\substack{j=0 \\ 0<j+k}}^{p} \sum_{\substack{k=0 \\ (j-k)/N \in \mathbb{Z}}}^{q} \binom{p}{j}\binom{q}{k} \frac{c_{jk} + c_{kj} e^{i2\theta(j-k)}}{2c_{00}} D_{p-j,q-k}^{N}$$

The axis orientation must be known and fixed

Blur-invariant phase correlation

- No landmarks
- Fast and robust image registration
- Preprocessing for MBD and SR

Phase correlation of the primordial images

$$C(\mathbf{u}) = \frac{I_N^{(f)} I_N^{(g)*}}{\left| I_N^{(f)} \right| \left| I_N^{(g)} \right|}$$

$$C_j(\mathbf{u}) = \frac{K_{N,j}^{(f)} K_{N,j}^{(g)*}}{|K_{N,j}^{(f)}||K_{N,j}^{(g)}|} = \frac{K_{N,j}^{(f)} K_{N,j}^{(g)*}}{|K_{N,j}^{(g)}|^2} = \frac{K_{N,j}^{(f)}}{K_{N,j}^{(f(\mathbf{x}-\boldsymbol{\Delta}))}} = e^{-i\mathbf{u}\cdot(\mathrm{R}_\mathrm{j}^\mathrm{T}\boldsymbol{\Delta}-\boldsymbol{\Delta})}$$

Moment-based focus measure

- Odd-order moments → blur invariants
- Even-order moments → blur/focus measure

$$M(g) = \mu_{20}^{(g)} + \mu_{02}^{(g)} = (\mu_{20}^{(f)} + \mu_{02}^{(f)}) + \mu_{00}^{(f)}(\mu_{20}^{(h)} + \mu_{02}^{(h)})$$

If $M(g1) > M(g2)$ → $g2$ is less blurred

(more focused)

Other focus measures

- Gray-level variance
- Energy of gradient
- Energy of Laplacian
- Energy of high-pass bands of WT

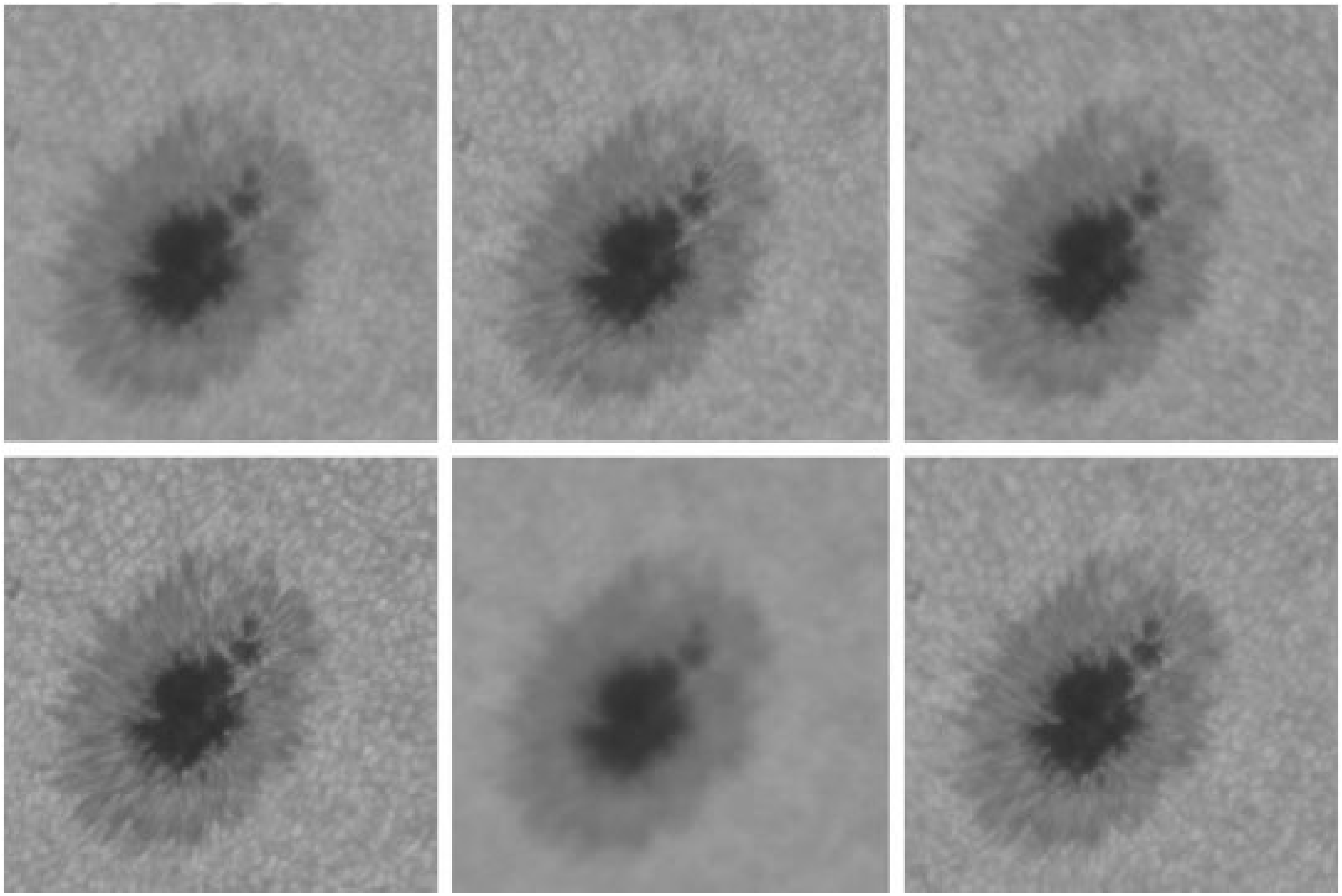
Desirable properties of a focus measure

- Agreement with a visual assessment
- Unimodality
- Robustness to noise
- Robustness to small image changes

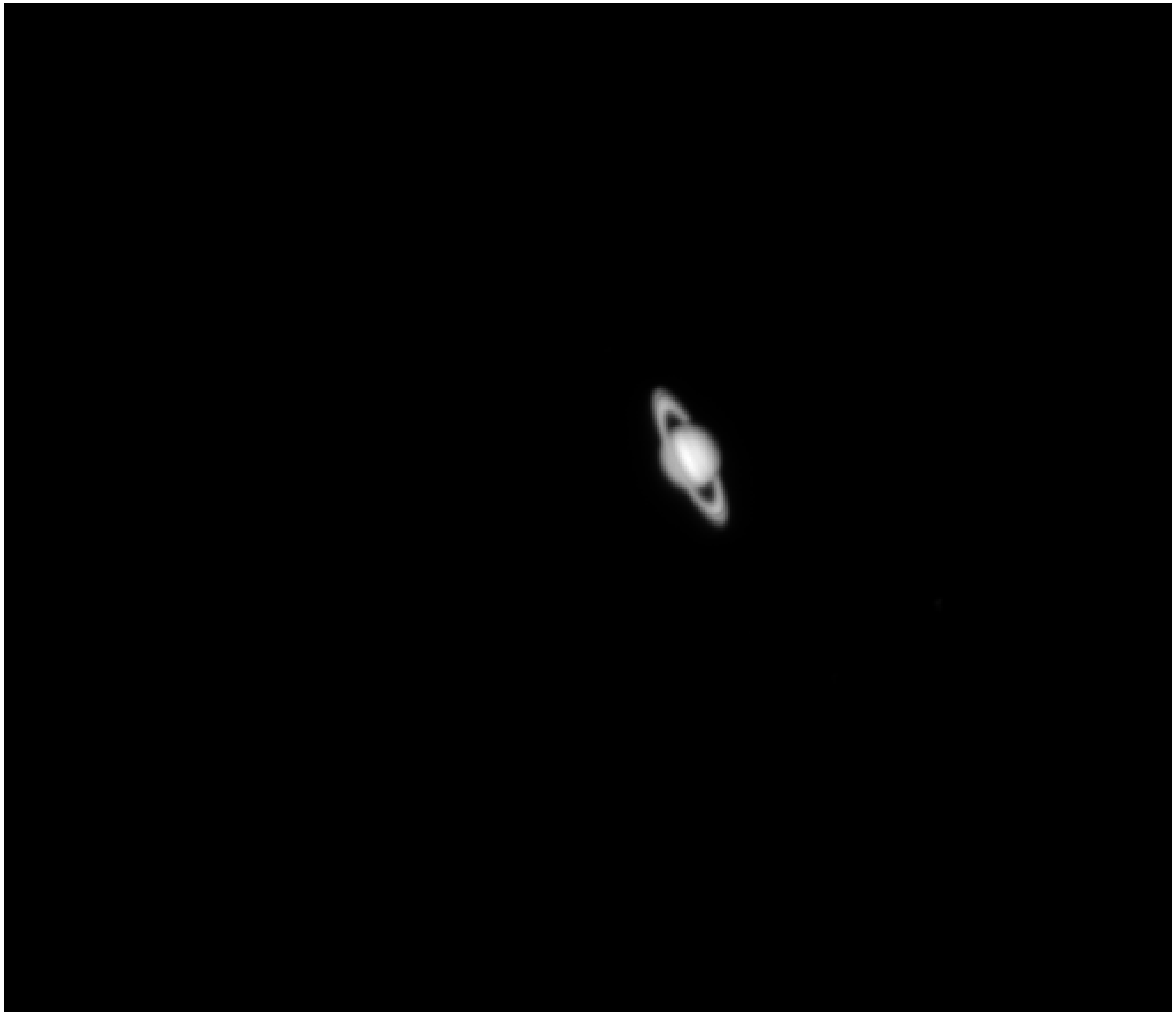
Agreement with a visual assessment

Robustness to noise

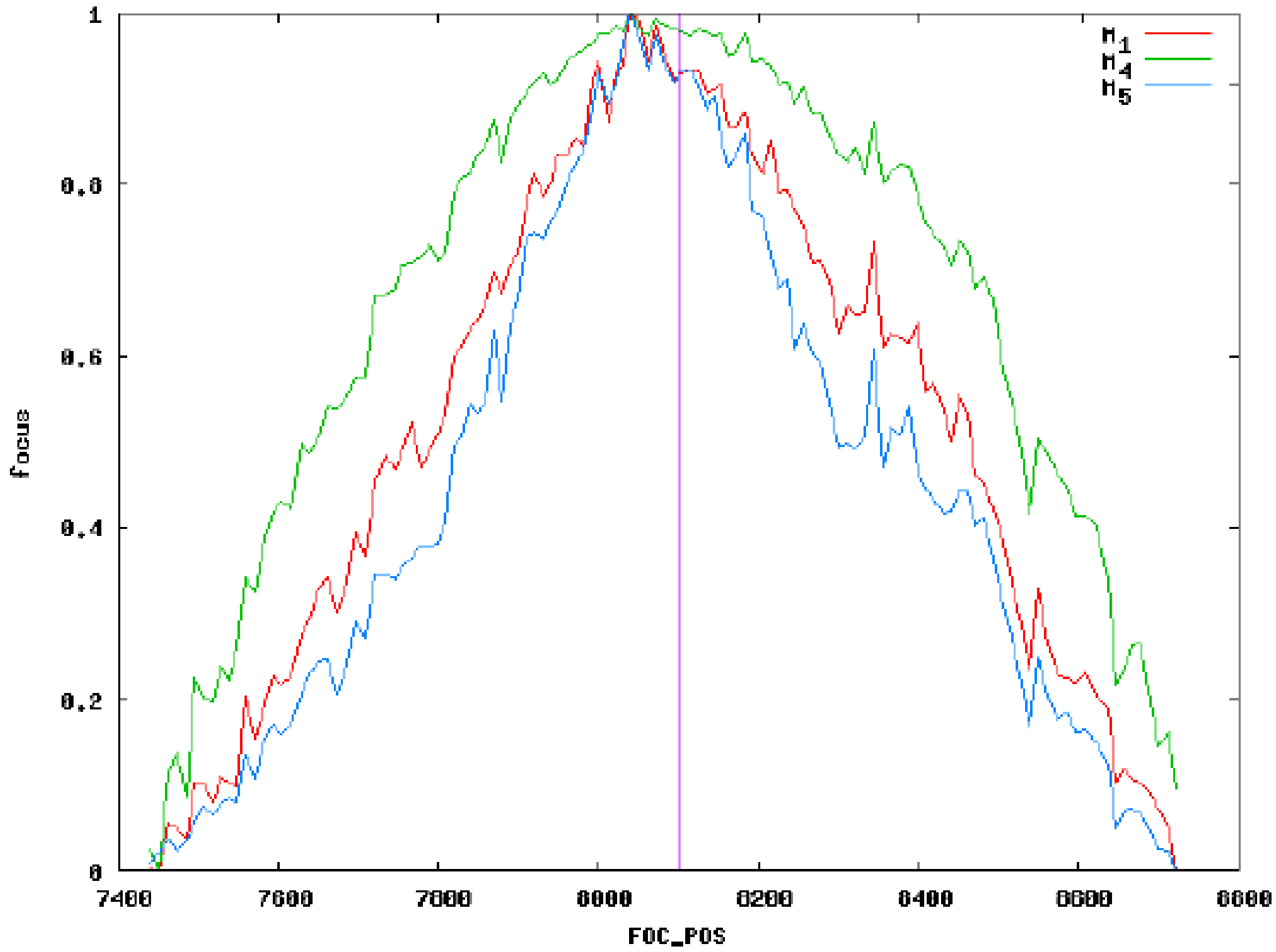
Sunspots – atmospheric turbulence



Saturn images



Saturn images – comparing focus measures



Moments are generally worse than wavelets and other differential focus measures because they are too sensitive to local changes but they are very robust to noise.

Multifocus fusion based on a local blur measurement