

Speciální funkce a transformace ve zpracování obrazu

1.část

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<http://zoi.utia.cas.cz/PGR013/materialy>

- Fourierova transformace
- wavelety: něco málo teorie
- wavelety: aplikace v DZO

Fourierova transformace

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i k x} dx.$$

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{2\pi i k x} dk$$

oscilační X úhlová frekvence

$$\omega \equiv 2\pi k$$

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-iyt} dt$$

$$\mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt$$

$$\mathcal{F}^{-1}[g(y)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)e^{iyt} dy$$

$$\mathcal{F}^{-1}[H(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{i\omega t} d\omega.$$

Vlastnosti

- **linearita**

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)] = aF(k) + bG(k).$$

- **konvoluce**

$$\mathcal{F}[f]\mathcal{F}[g] = \mathcal{F}[f * g]$$

convolution theorem

- **posun**

$$\mathcal{F}_x[f(x - x_0)](k) = e^{-2\pi i k x_0} F(k).$$

shift theorem

- **rotace**

$$F(R(f)) = R(F(f))$$

- **změna měřítka**

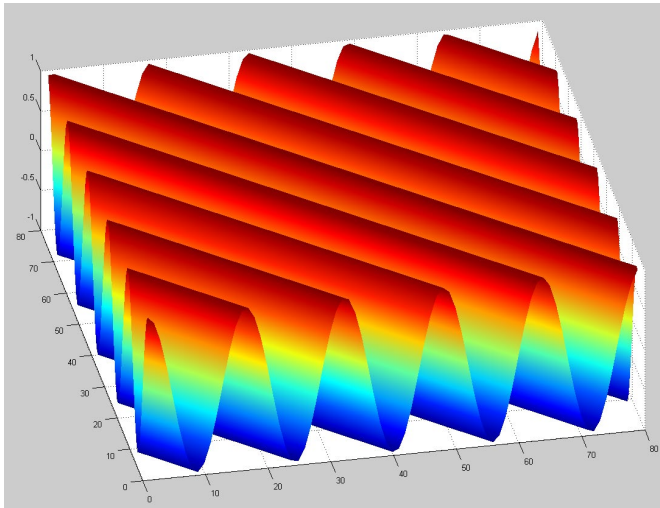
$$\mathcal{F}_x[f(ax)](k) = |a|^{-1} F\left(\frac{k}{a}\right).$$

similarity theorem

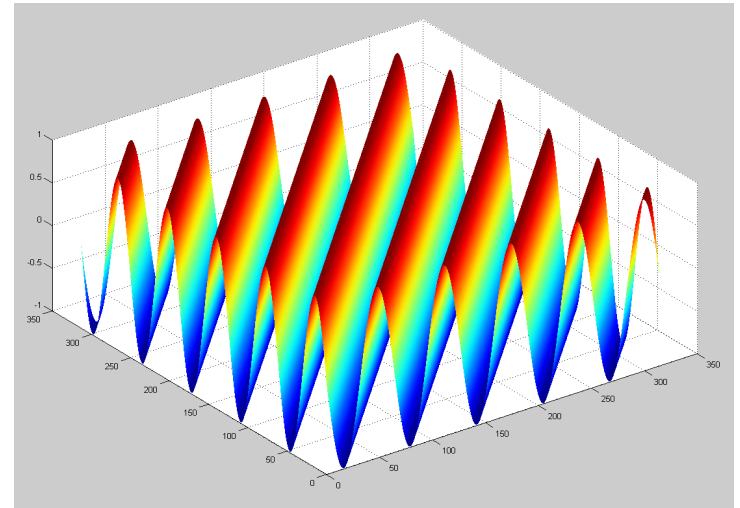
Fourierova transformace - 2D

$$F(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(k_x, k_y) e^{-2\pi i(k_x x + k_y y)} dk_x dk_y$$

$$f(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) e^{2\pi i(k_x x + k_y y)} dx dy.$$

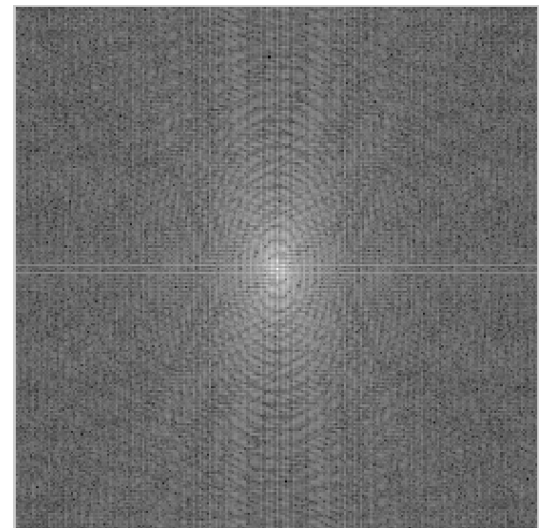
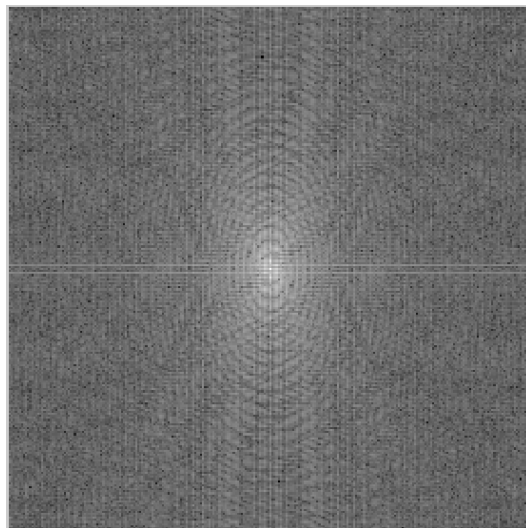
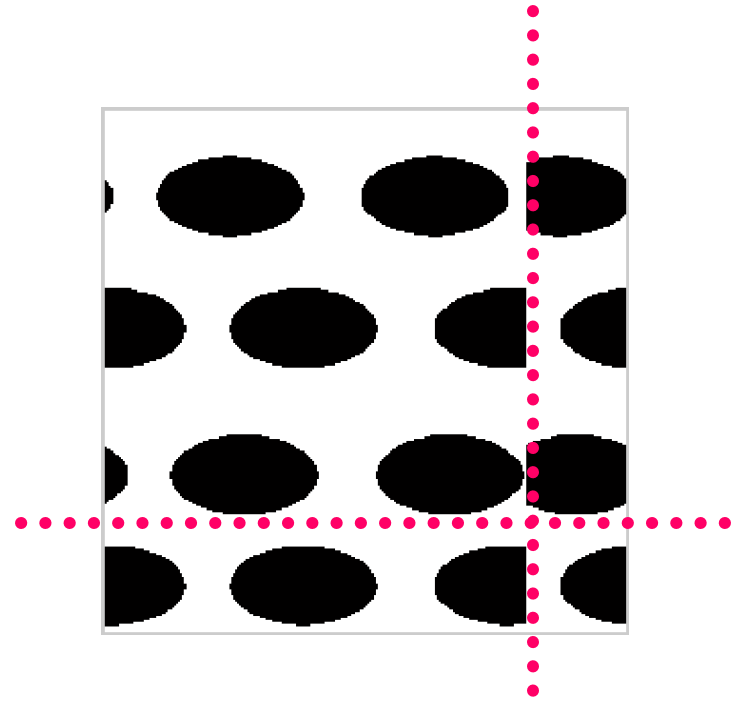
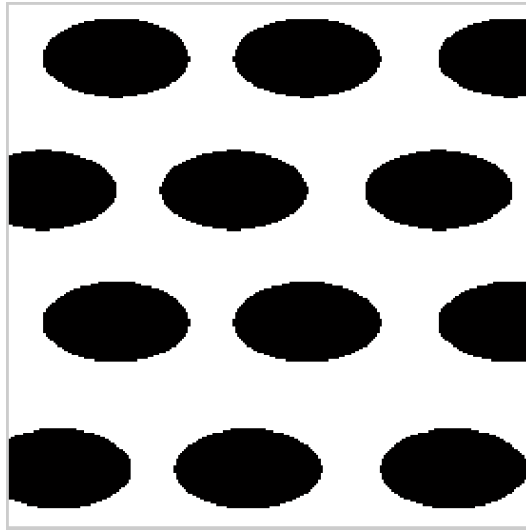


real, $u=v$



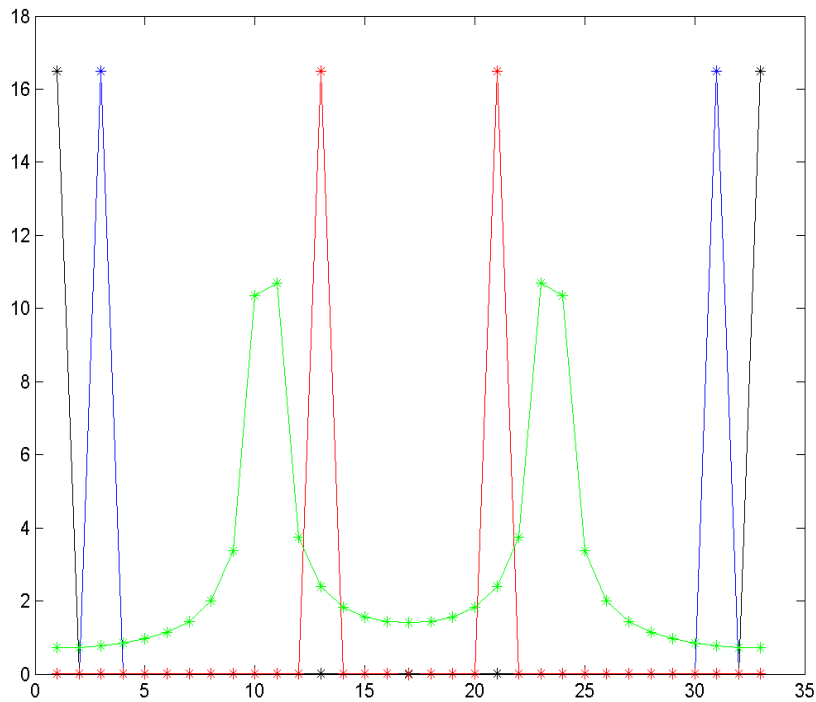
imag, $u=v$

Periodické prodloužení



$$F_n \equiv \sum_{k=0}^{N-1} f_k e^{-2\pi i n k / N}.$$

$$f_k \equiv \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{2\pi i n k / N}.$$



4 / N

6.5 / N

17 / N

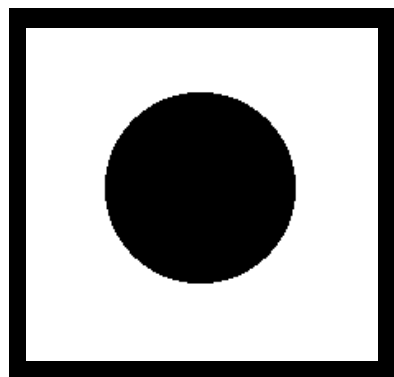
19 / N

Použití

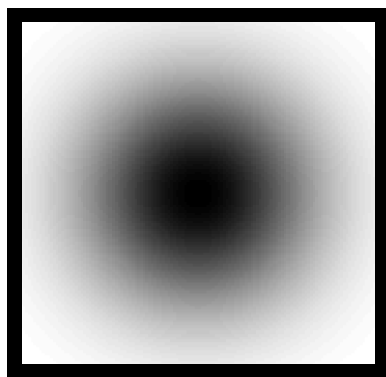
- filtrace
- registrace
- interpolace
- reprezentace objektů
- reprezentace textur

Filtrace ve frekvenční oblasti

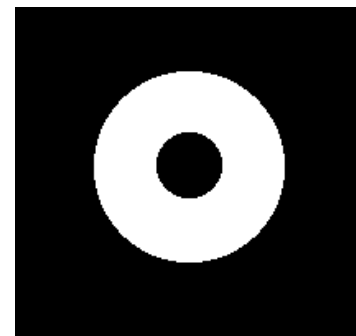
$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$$



high pass

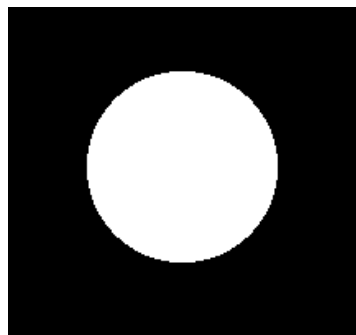


Gaussian high pass

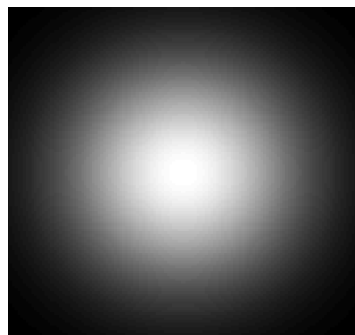


band pass

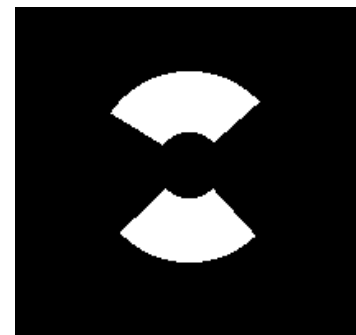
low pass



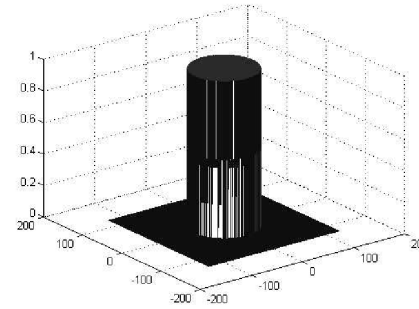
Gaussian low pass



directional



Low pass step filtr



LPF image, $r_0 = 36$



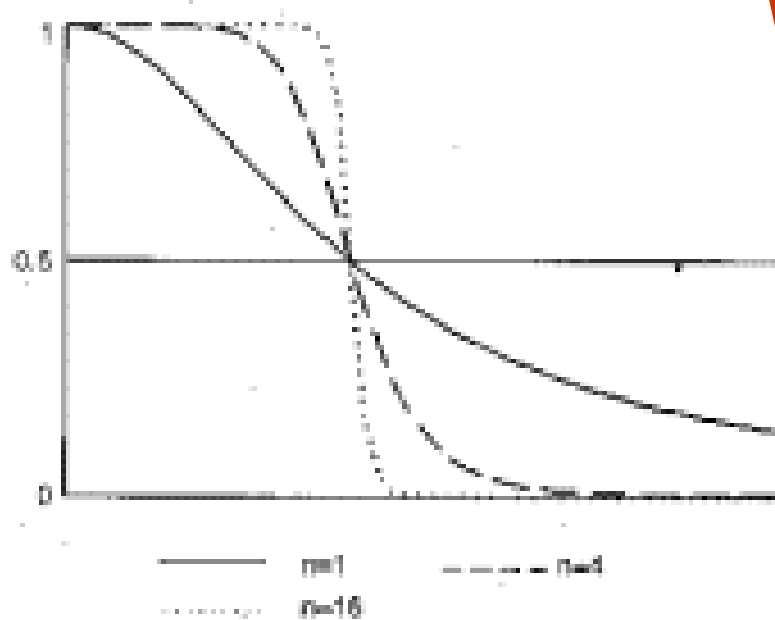
LPF image, $r_0 = 26$

Butterworth filtr

low pass

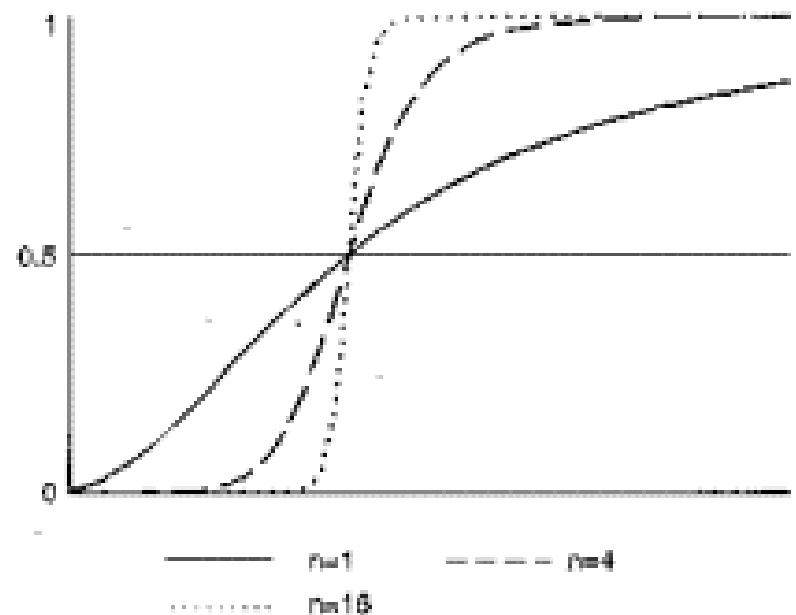
$$H(u, v) = \frac{1}{1 + \left[\frac{D(u, v)}{D_0} \right]^{2n}}$$

$$D(u, v) = \sqrt{u^2 + v^2}$$



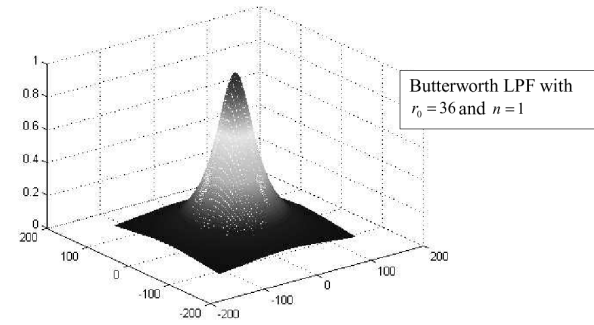
high pass

$$H(u, v) = \frac{1}{1 + \left[\frac{D_0}{D(u, v)} \right]^{2n}}$$



n= 1, 4, 16

Butterworth filtr



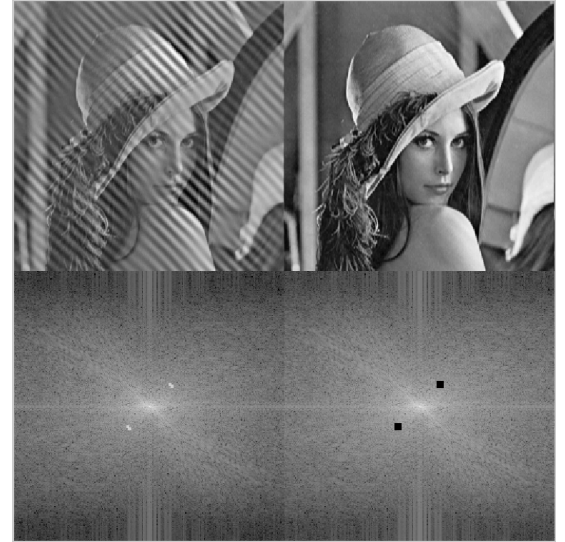
LPF image, $r_0 = 13$



LPF image, $r_0 = 10$

- **Filtrace periodického poškození**

- **Inverzní filtrace**



- **Známý typ PSF**

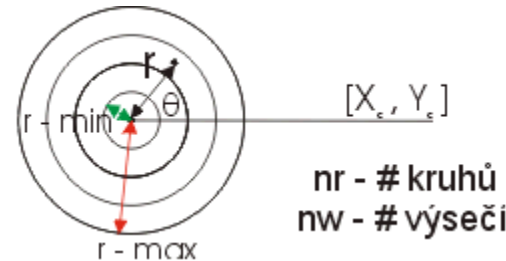
Registrace - fázová korelace

- kros korelace + $\mathcal{F}_x[f(x - x_0)](k) = e^{-2\pi i k x_0} F(k).$

$$\frac{F(\text{Image}(x,y)) \cdot F^*(\text{Window}(x,y))}{|F(\text{Image}(x,y)) \cdot F^*(\text{Window}(x,y))|} = e^{2\pi i (x T_X + y T_Y)}$$

SPOMF symmetric phase - only matched filter

Log-polar transformace



- polar

$$r = [(x - x_c)^2 + (y - y_c)^2]^{1/2}, \quad \theta = \tan^{-1} \left(\frac{y - y_c}{x - x_c} \right)$$

- log

$$R = \frac{(n_r - 1) \log (r / r_{min})}{\log (r_{max} / r_{min})}, \quad W = \frac{n_w \theta}{2\pi}$$

RTS registrace

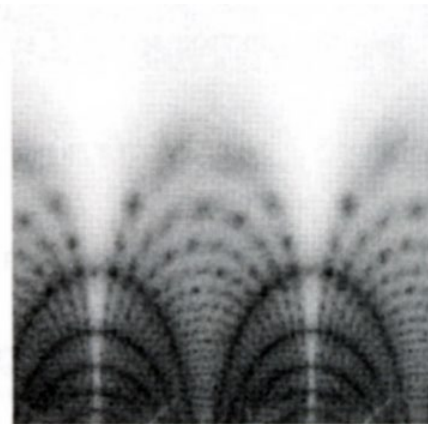
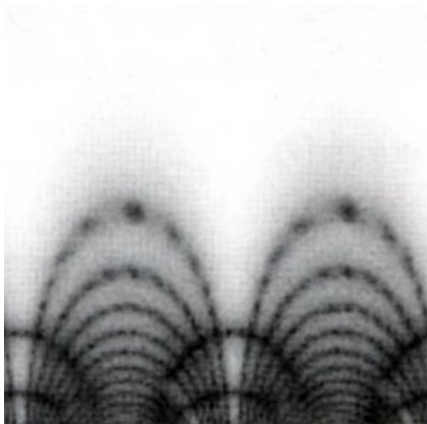
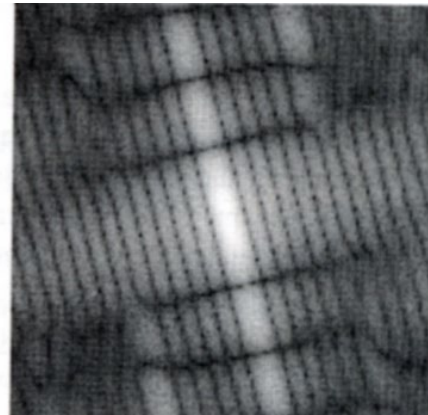
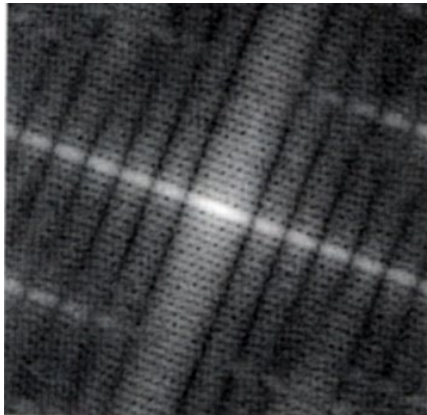
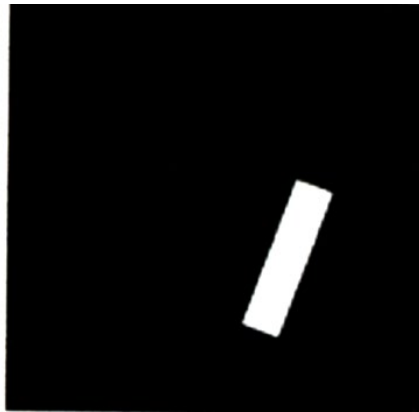
$$\mathcal{F}_x[f(x - x_0)](k) = e^{-2\pi i k x_0} F(k).$$

$$F(R(f)) = R(F(f))$$

$$\mathcal{F}_x[f(ax)](k) = |a|^{-1} F\left(\frac{k}{a}\right).$$

FT \longrightarrow | | \longrightarrow log-polar \longrightarrow FT \longrightarrow fázová korelace

- π - periodicitu amplitudy - > 2 úhly
- $\log(\text{abs}(\text{FT})+1)$
- problémy s diskretním prostředím



Fourierovy deskriptory



$$F_n \equiv \sum_{k=0}^{N-1} f_k e^{-2\pi i n k / N}$$

$$f(t) = x(t) + iy(t)$$

Posun

Rotace

Měřítko

Změna start bodu

- změna $F(0)$

- změna fáze

- vynásobení konstantou

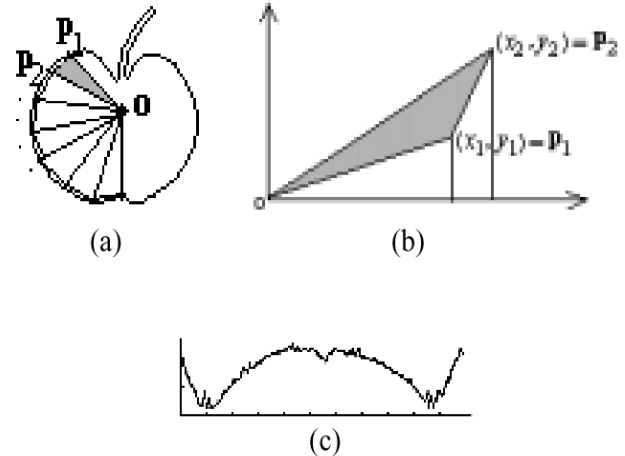
- posun v 1D reprezentaci

Fourierovy deskriptory

periodická funkce

- souřadnice $f(t) = x(t) + iy(t)$
- vzdálenost od těžiště $f(t) = ([x(t) - x_c]^2 + [y(t) - y_c]^2)^{1/2}$

- plocha



Fourierovy deskriptory - interpolace

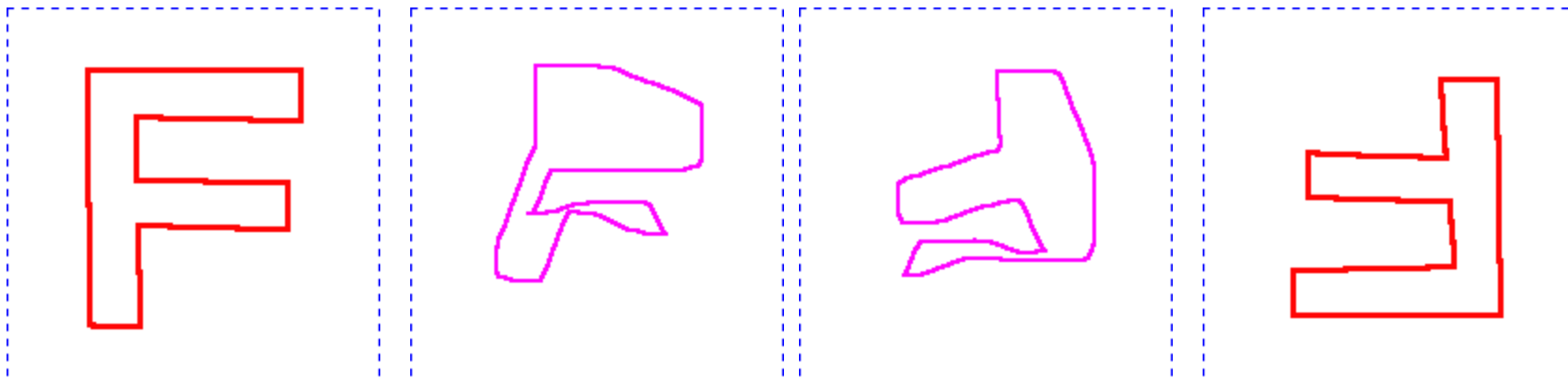
separace

- tvar

- pozice

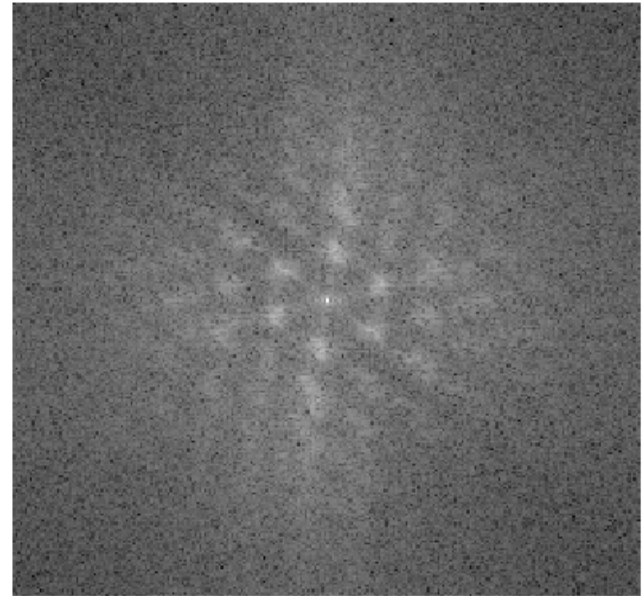
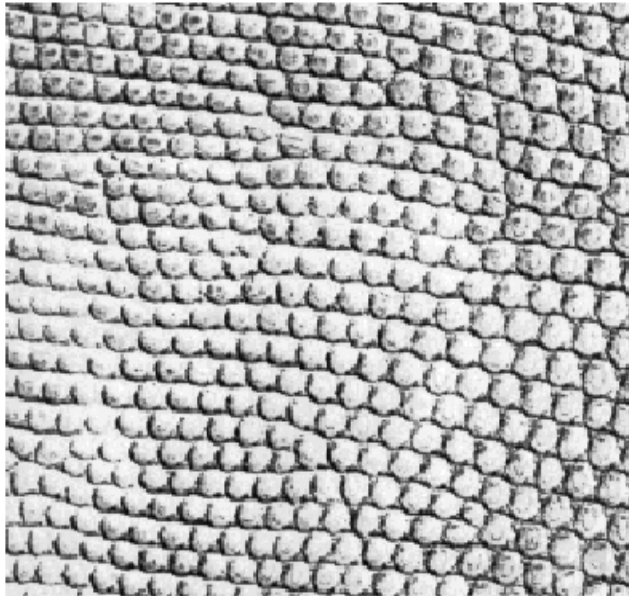
- měřítko

- orientace

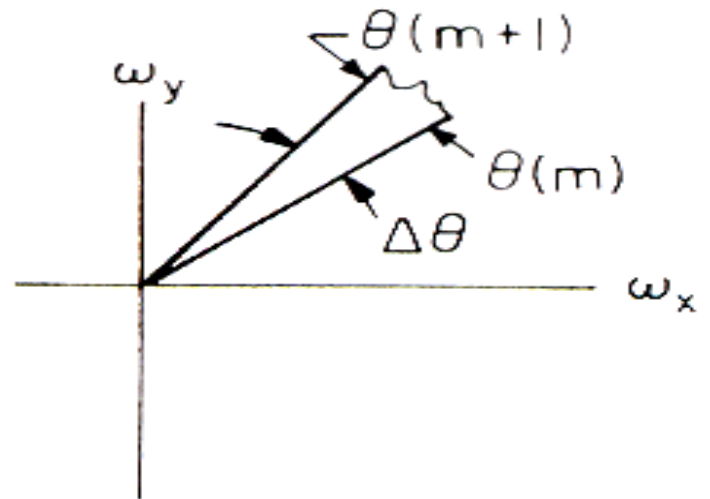
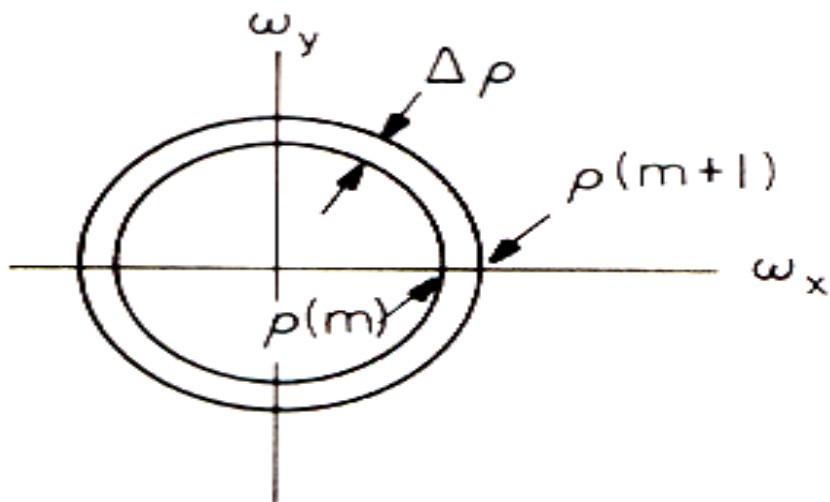
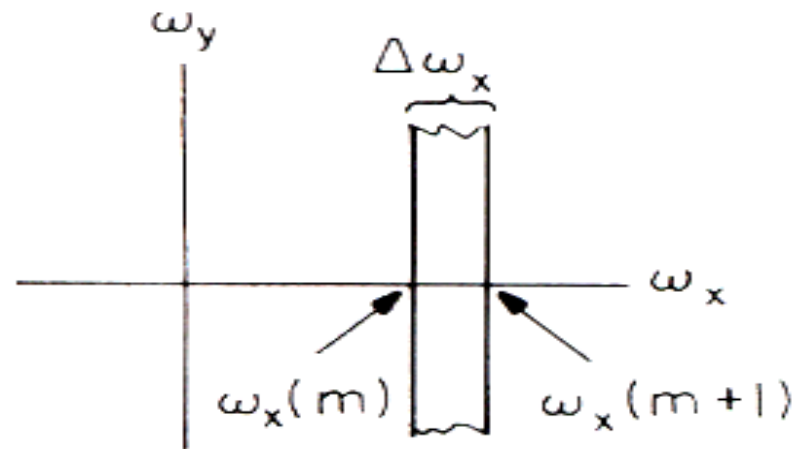
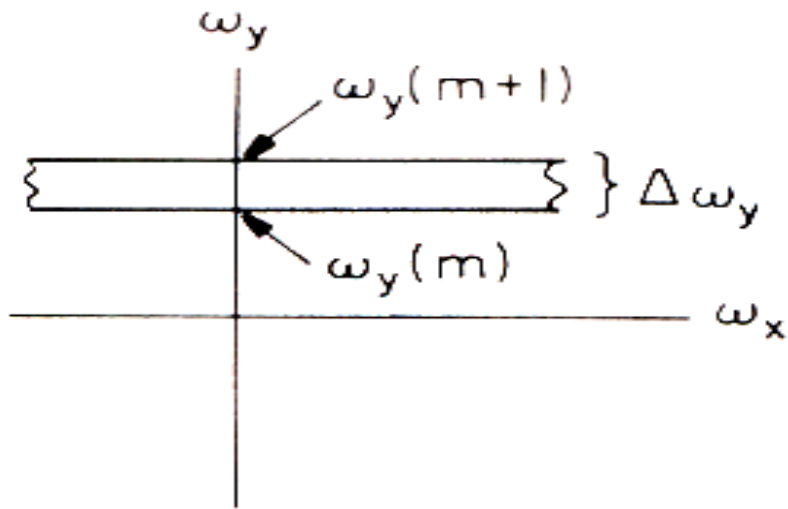


medicínské řezy

Textury - popis



Textury - popis

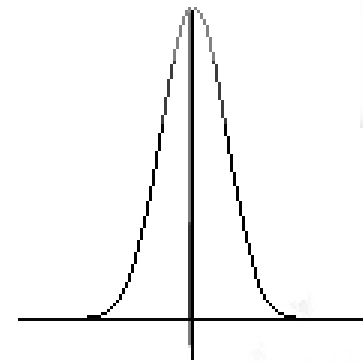


Fourierova transformace

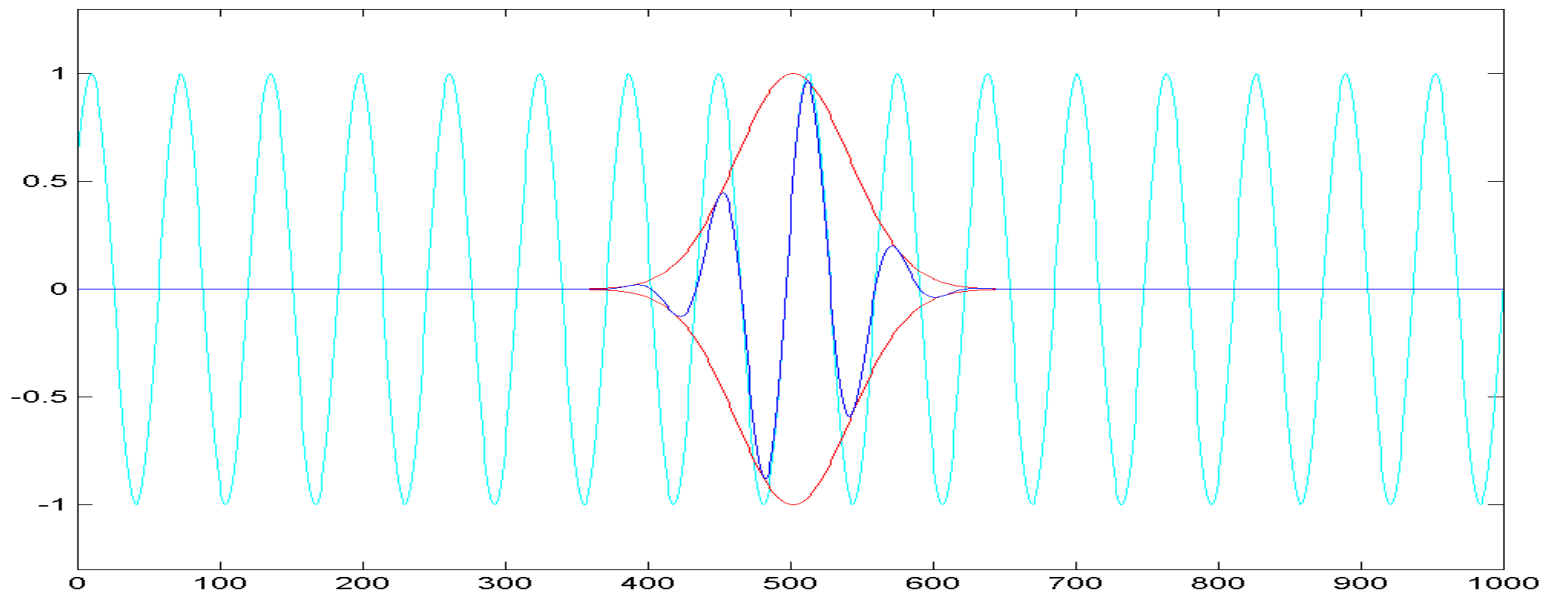
- $e^{j\theta} = \cos(\theta) + j \sin(\theta)$ - základní stavební prvky FT
- pro každou frekvenci – sinusoida dané frekvence porovnána se signálem
- obsahuje-li signál danou frekvenci – korelace je velká → velké FT koeficienty
- nemá-li signál žádnou část dané frekvence, korelace na dané frekvenci je malá/nulová → malý / nulový FT koeficient

Okénková Fourierova transformace

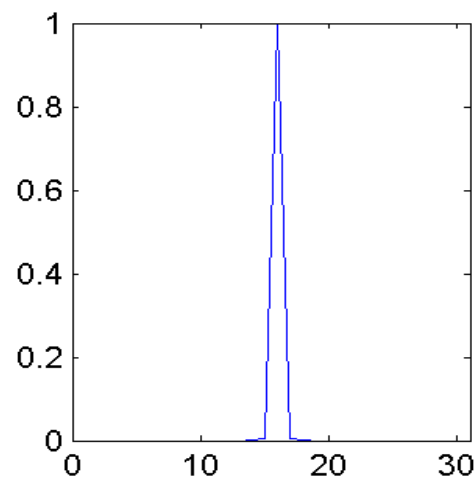
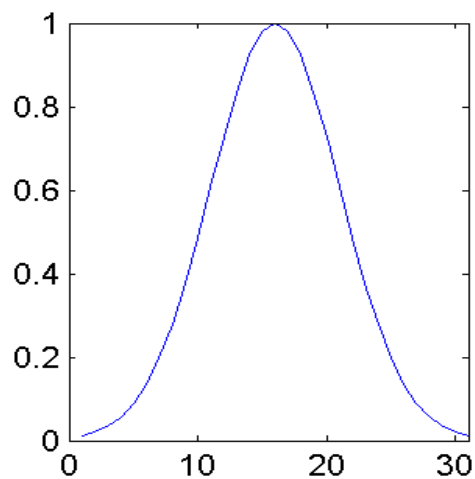
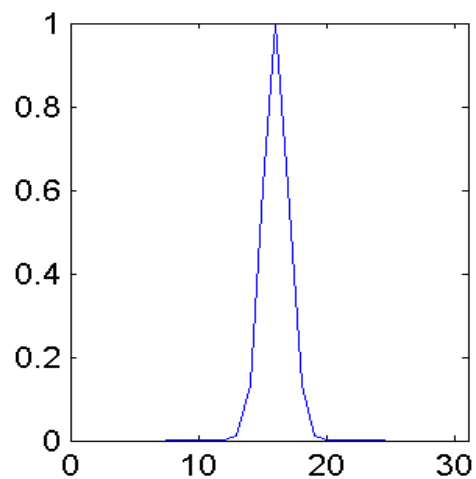
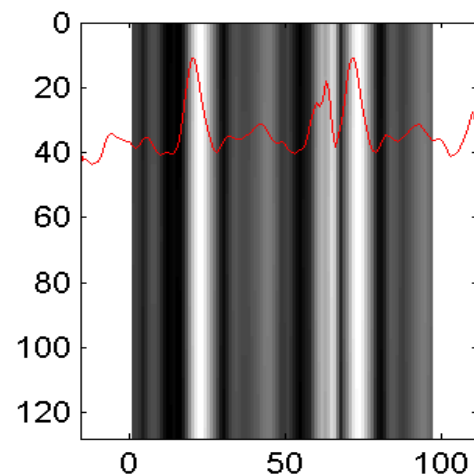
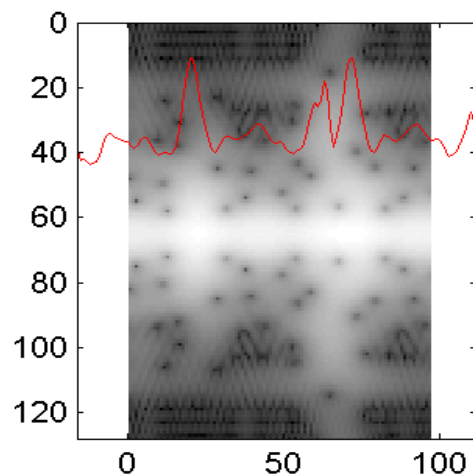
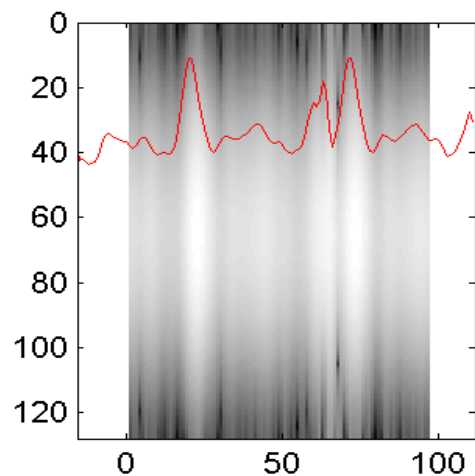
$$F_W(\tau, f) = \int_{-\infty}^{+\infty} f(t) w^*(t-\tau) e^{-i2\pi ft} dt$$



$$w(t) = e^{-\pi t^2}$$



Okénková Fourierova transformace



Okénková Fourierova transformace

- výpočet různých FT pro po sobě jdoucí časové intervaly
- time-frequency reprezentace
„1-D time domain“ → „2-D time-frequency“
- volba okna tvar, šířka
- šířka okna – signál v něm stacionární
- širší okno – menší „time“ rozlišení

$$\mathcal{F}_x[f(ax)](k) = |a|^{-1} F\left(\frac{k}{a}\right).$$

Volba okna

Dva extrémy

- $W(t) = 1$ $W(t)$ nekonečně široké → klasická FT

výborné „frequency“ rozlišení, žádná „time“ informace

- $W(t) = \delta(t)$ $W(t)$ nekonečně úzké → konstanta

výborné „time“ rozlišení, žádná „frequency“ informace

Okno zvoleno – rozlišení nastaveno v obou oblastech

Gaussovské okno – nejmenší

Heisenbergův princip

$$\Delta t * \Delta f \geq 1/(4 \pi)$$

Gaborův princip neurčitosti

Δt **Time rozlišení:**

separace 2 „špicí“ v časové oblasti

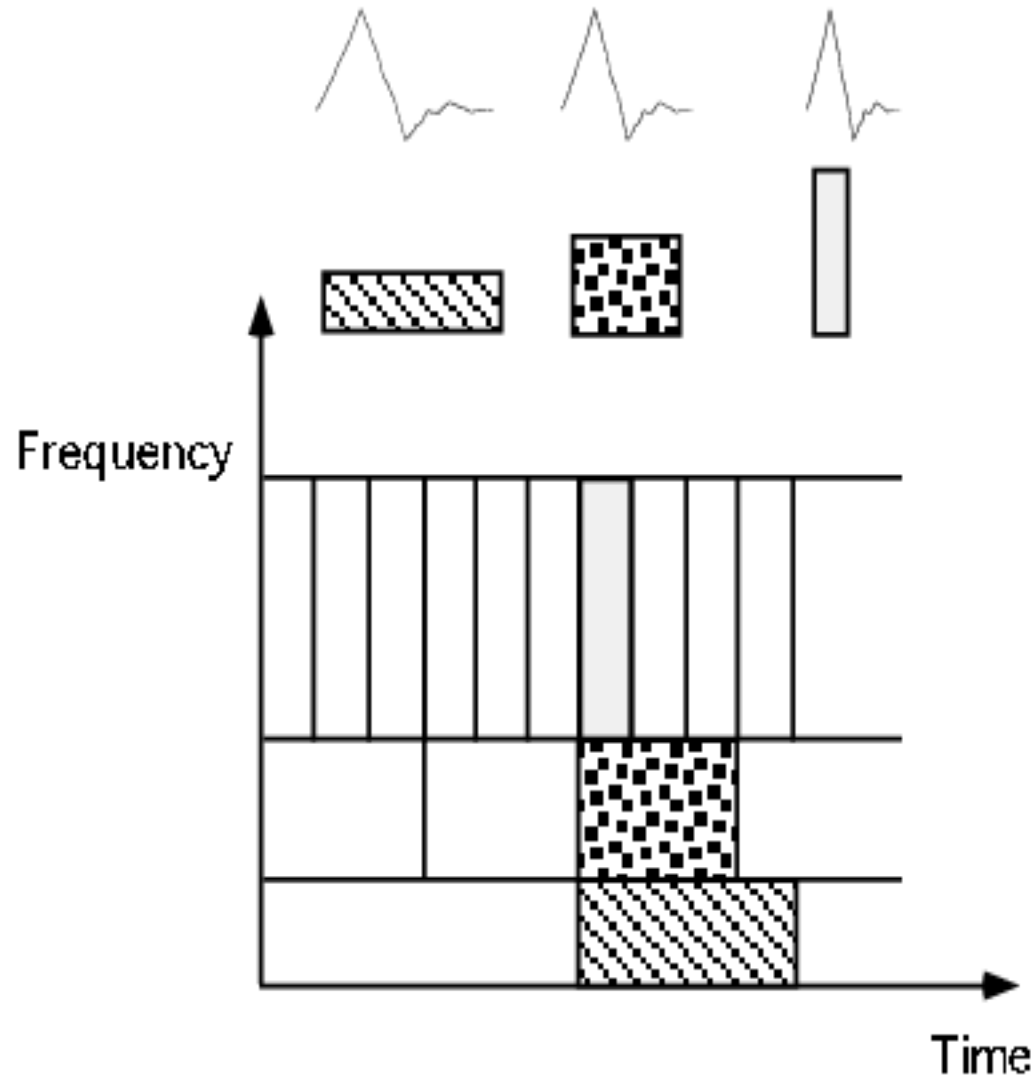
Δf **Frequency rozlišení :**

separace 2 spektrálních komponent

Obě rozlišení nemohou být libovolně velké!

Pouhé intervaly!!

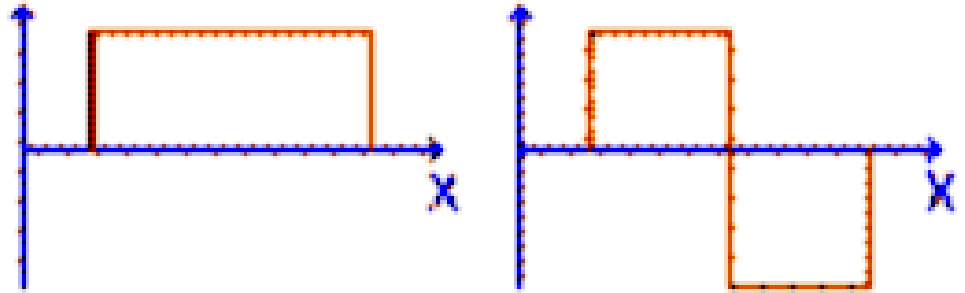
FT versus wavelets - plocha



Waveletová transformace

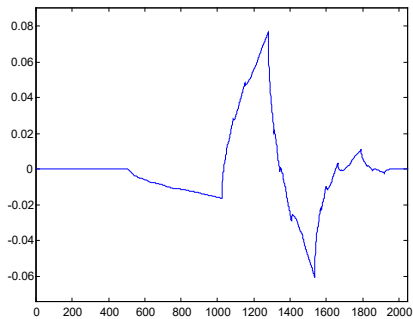
- **Základy teorie**
- **Aplikace**

Historie Wavelet



- 1909 Alfred Haar - **Haar** báze.
- 1946 **Gabor** - ne-orthogonální neomezené wavelety
- 1976 Croisier, Esteban a Galand - **filter banks** pro dekompozici a rekonstrukci signálu
- 1982 Jean **Morlet** použil Gabor wavelety k modelování seismických signálů

- 1987-1993 Stephane Mallat a Yves Meyer - **multiresolution** ,
Diskrétní Waveletová Transformace



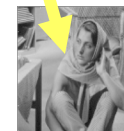
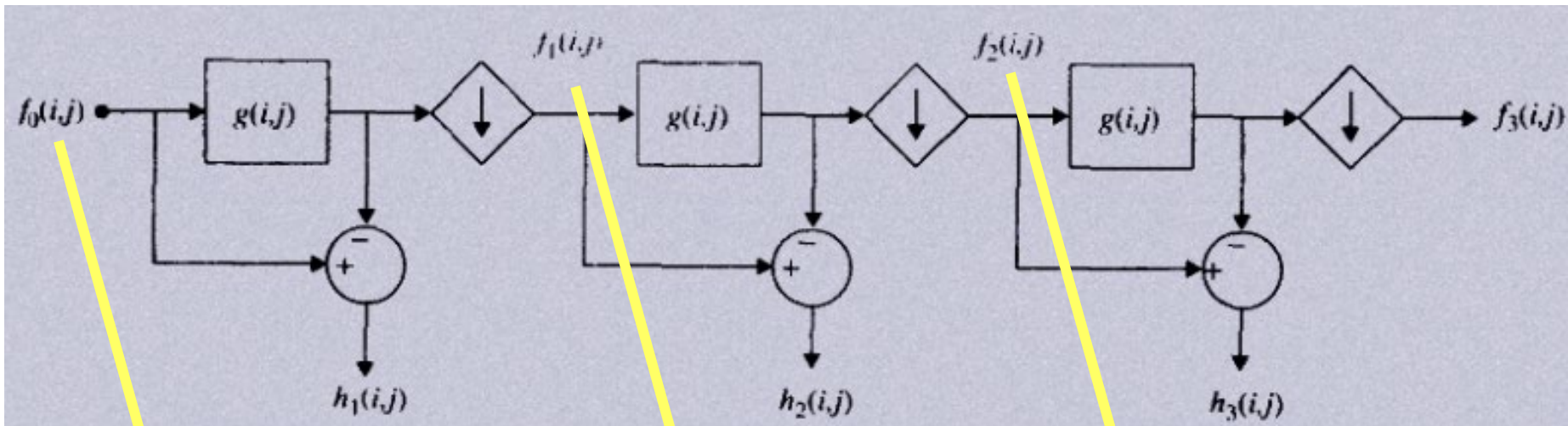
- 1988 Ingrid **Daubechies** - ortonormální, kompaktní skupina wavelet

Aplikace wavelet

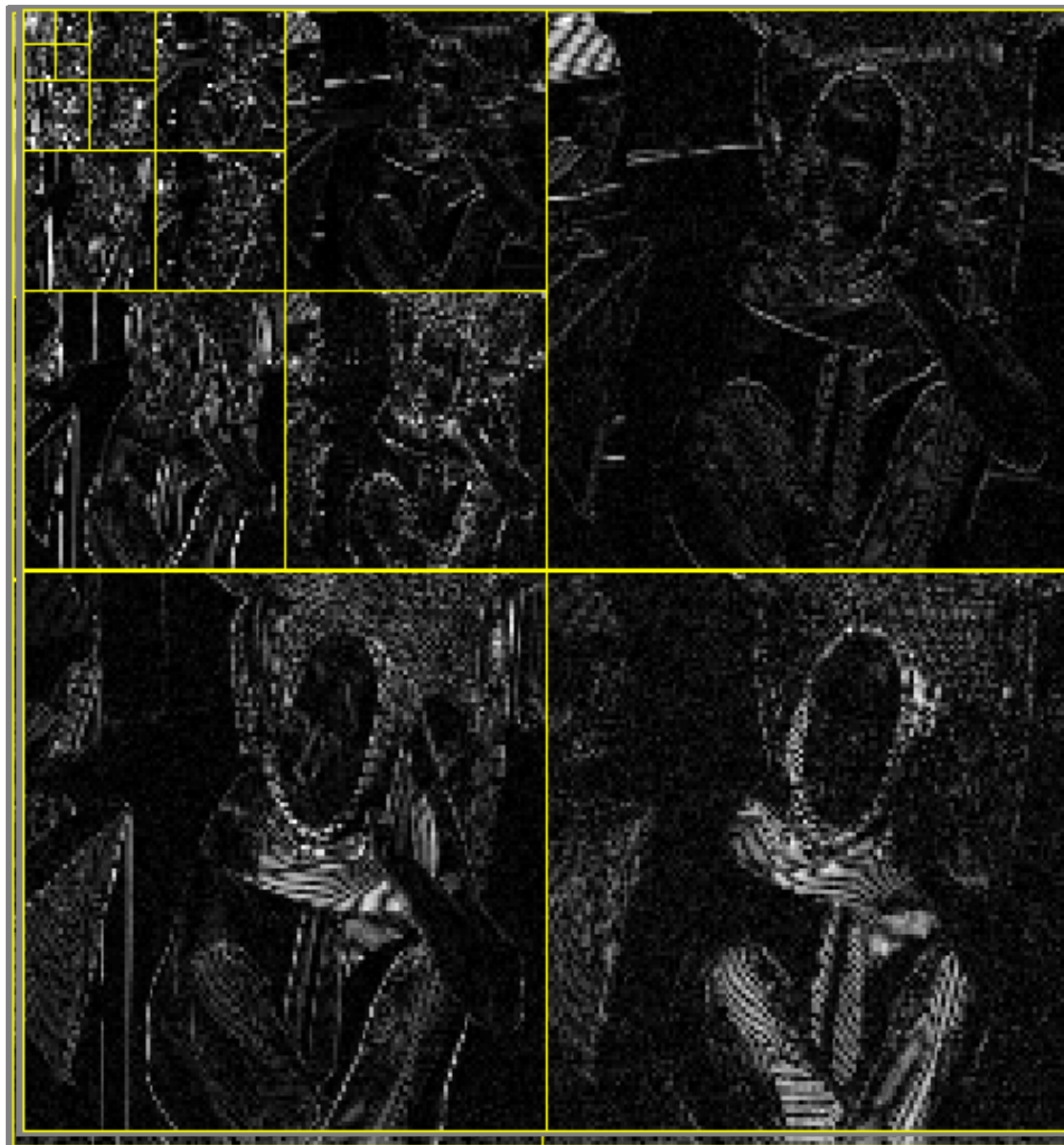
- **Komprese**
- **Problematika rozmazání**
- **Odstraňování šumu a poškození**
- **Detekce struktur**
- **Registrace**
- **Fúze dat s různým rozlišením**
- **Reprezentace**

O co tady jde ?

„Laplacian“ pyramida

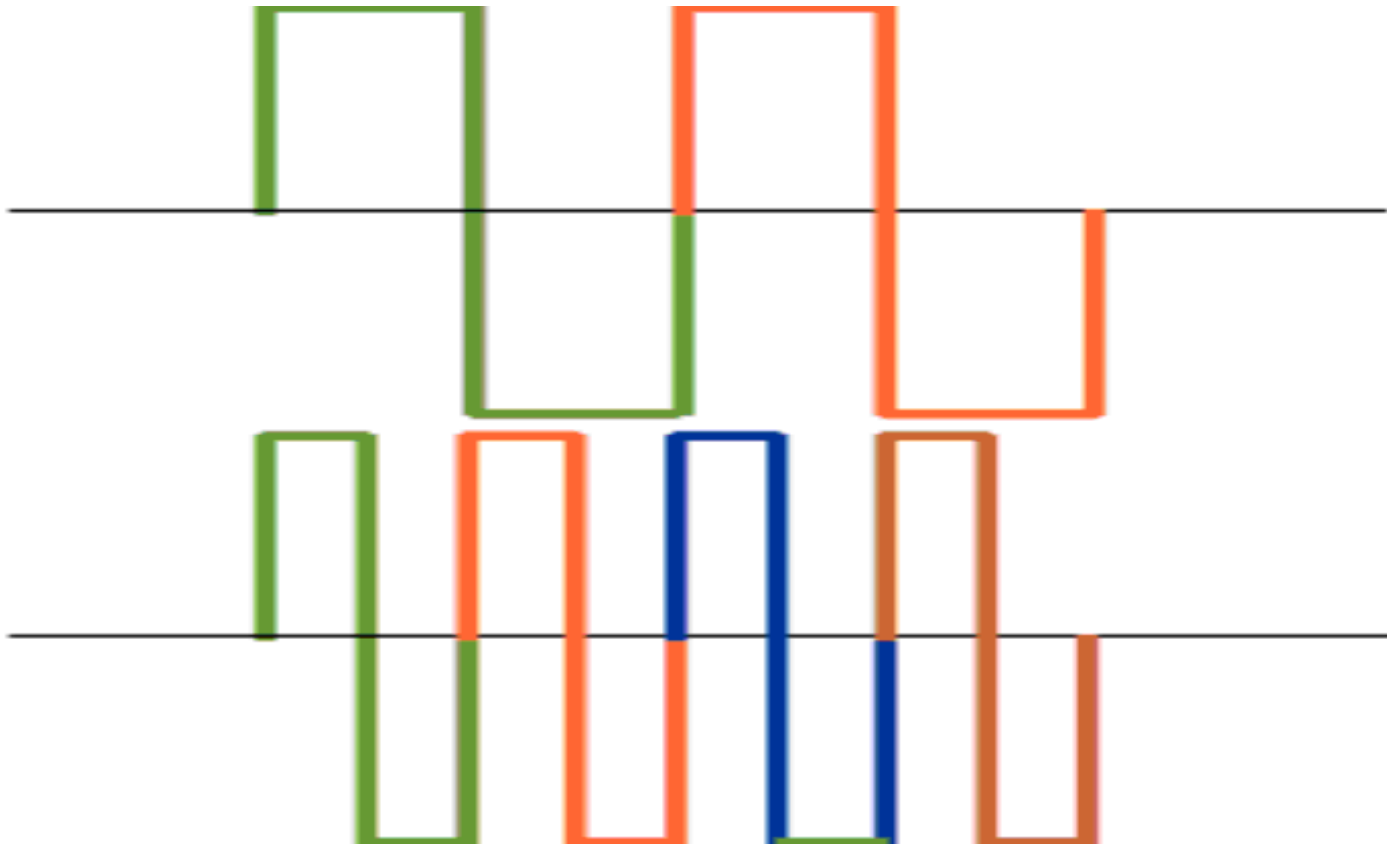
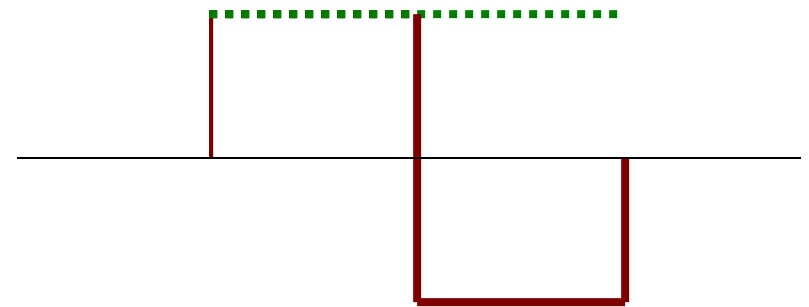


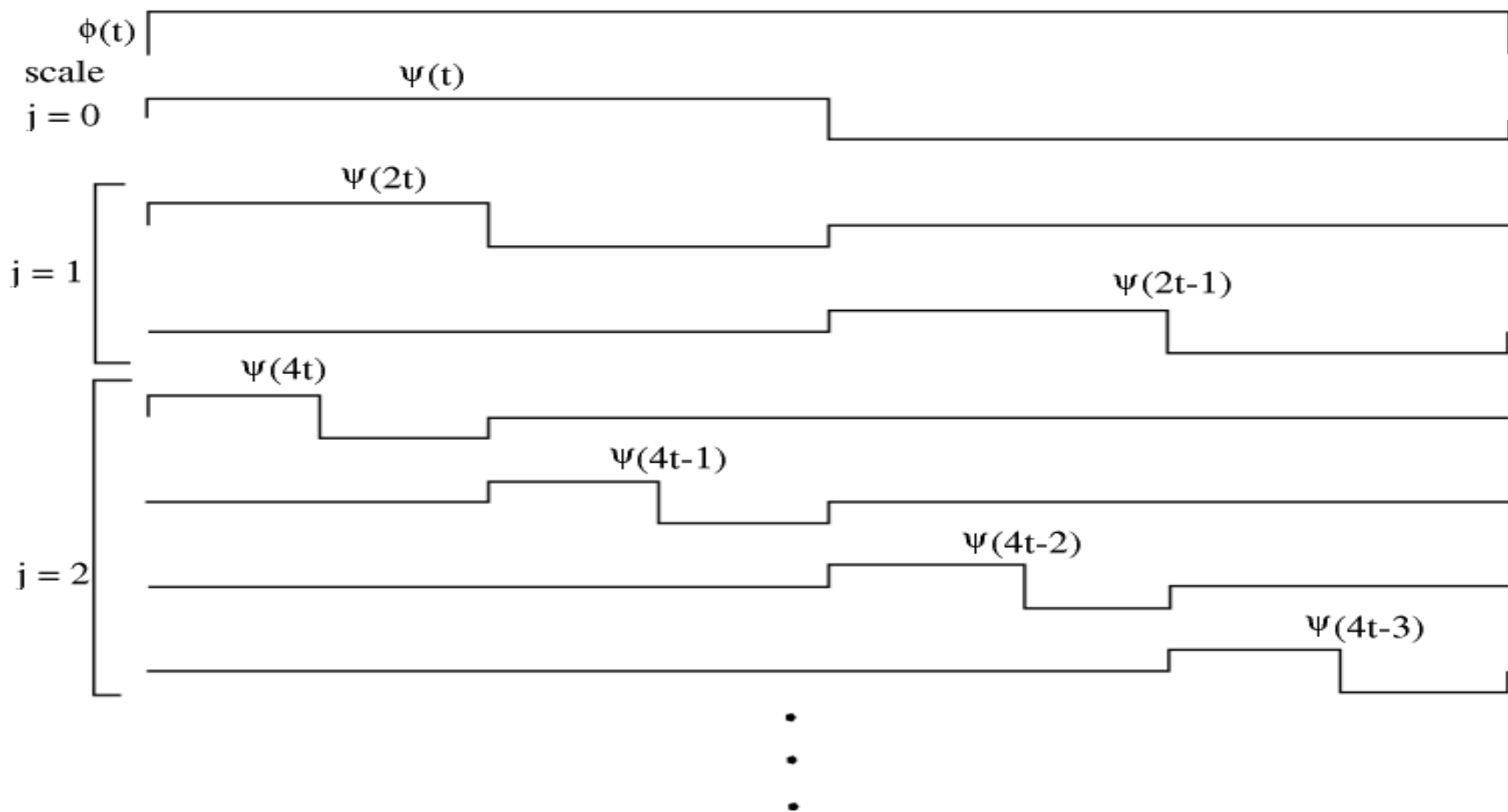
K čemu směřujeme ?



Haarova waveleta

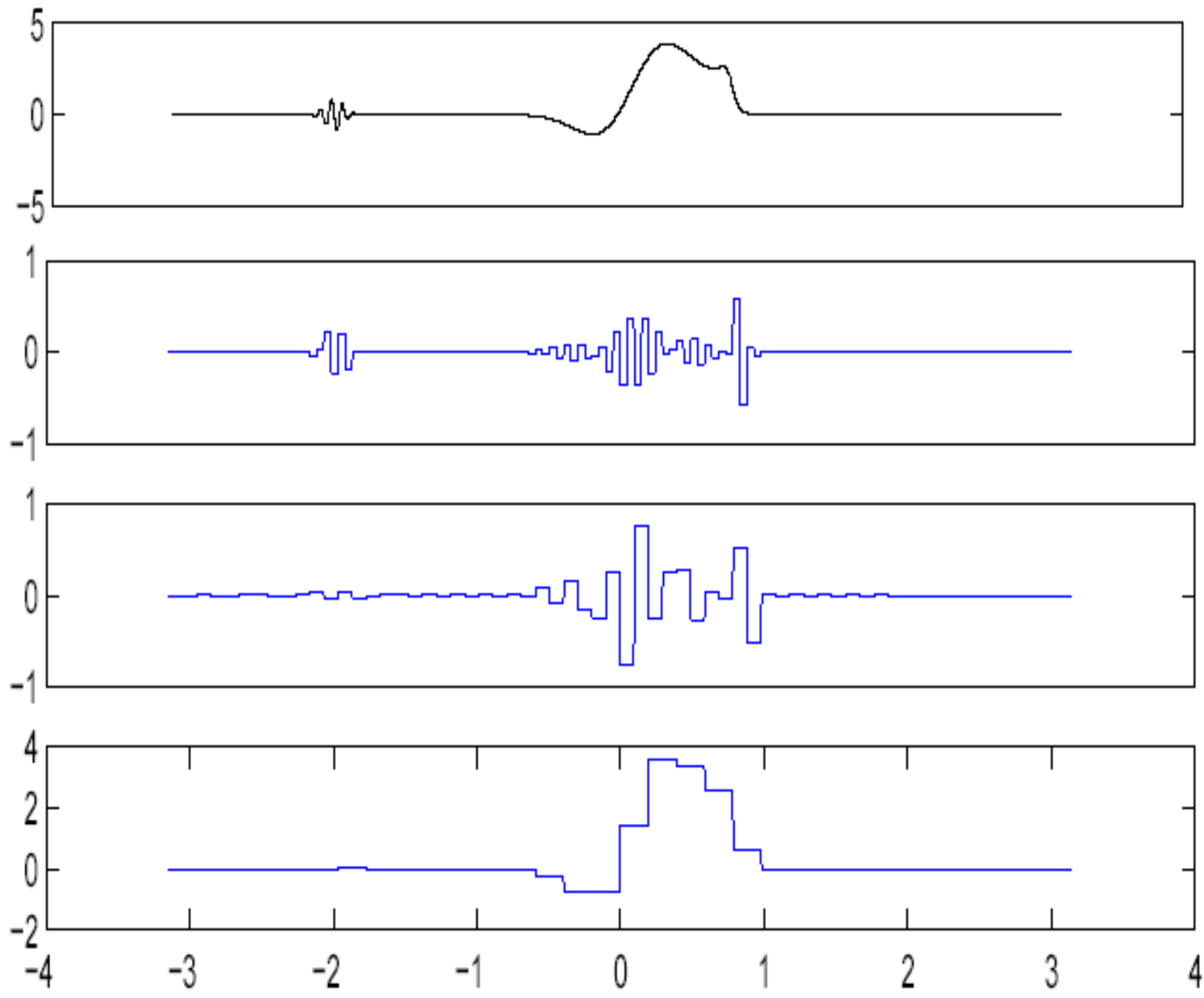
$$w(t) = \begin{cases} 1 & 0 \leq t \leq 1/2 \\ -1 & 1/2 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

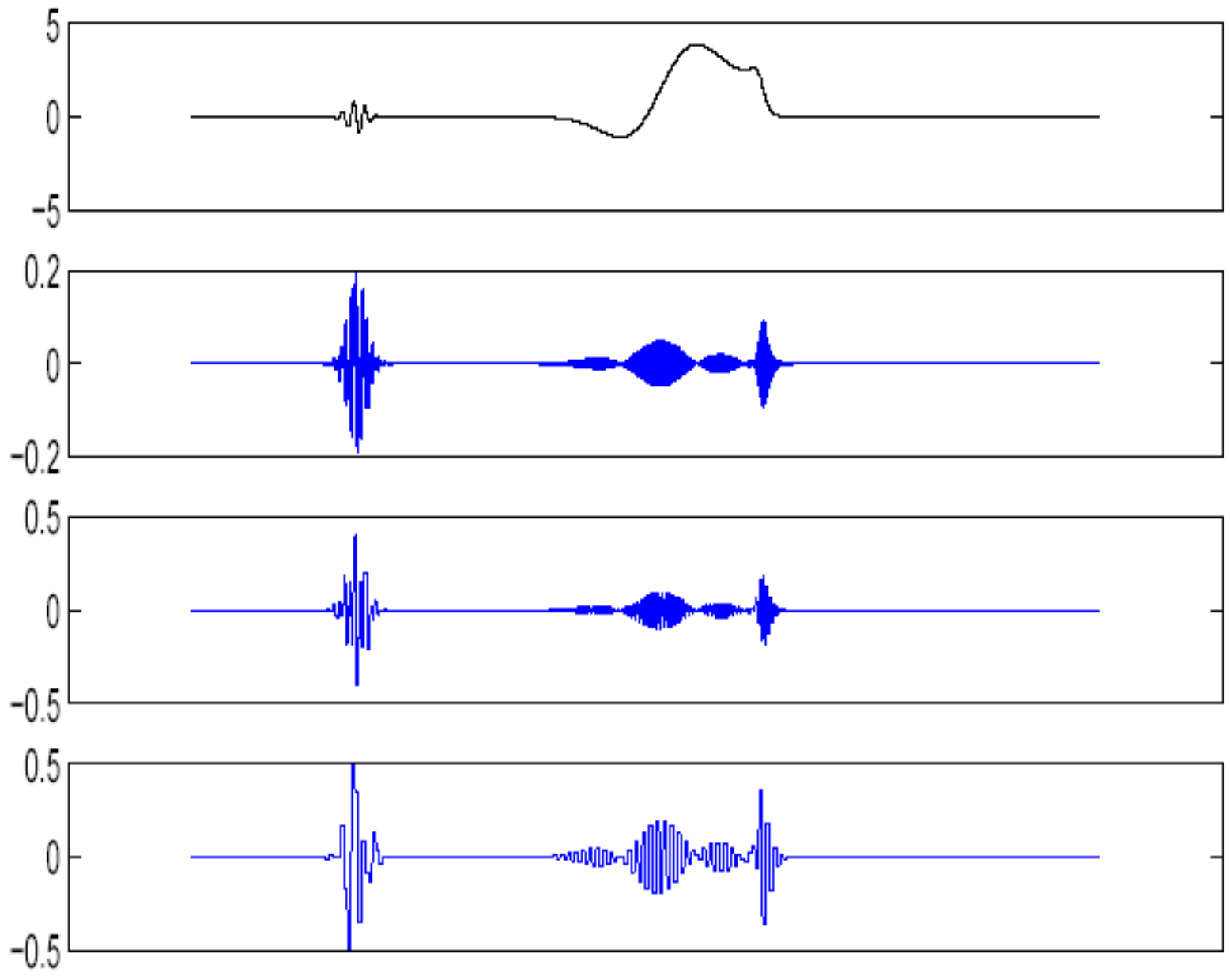




- kompaktní
- dyadická
- ortonormální

Figure 1.1: A signal and its derivatives.





$$\underline{y} = c^{(3)}$$

1	0	-3	2	1	0	1	2
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$$\underline{d}^{(2)}$$

$\frac{1}{\sqrt{2}}$	$-\frac{5}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
----------------------	-----------------------	----------------------	-----------------------

$$\underline{c}^{(2)}$$

$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$
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$$\underline{d}^{(1)}$$

1	-1
---	----

$$\underline{c}^{(1)}$$

0	2
---	---

$$\underline{d}^{(0)} \quad \boxed{-\sqrt{2}}$$

$$\underline{c}^{(0)} \quad \boxed{\sqrt{2}}$$

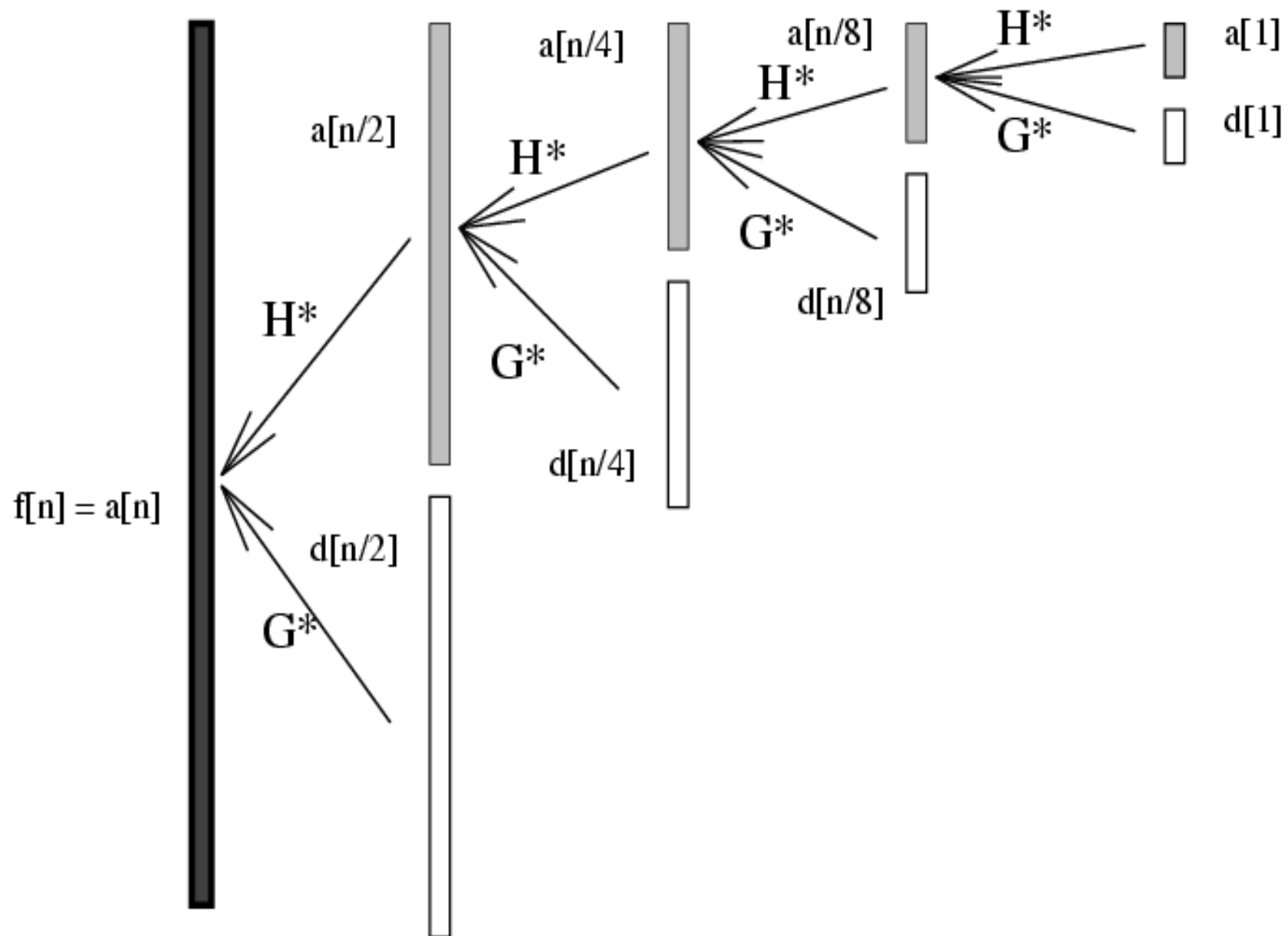
$\mathbf{h} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$ $\mathbf{g} = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]$
--

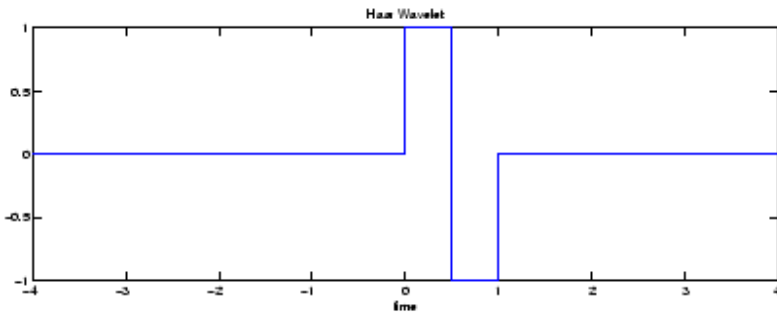
$$\begin{array}{l}
 \xi^{(0)} \quad \boxed{\sqrt{2}} \quad \longrightarrow \quad \boxed{1 \quad 1} \\
 \eta^{(0)} \quad \boxed{-\sqrt{2}} \quad \longrightarrow \quad \boxed{-1 \quad 1} \\
 \hline \hline
 \boxed{0 \quad 2}
 \end{array}$$

$$\begin{array}{l}
 \mathbf{h}^* = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right] \\
 \mathbf{g}^* = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]
 \end{array}$$

$$\begin{array}{l}
 \xi^{(1)} \quad \boxed{0 \quad 2} \quad \longrightarrow \quad \boxed{0 \quad 0 \quad \frac{2}{\sqrt{2}} \quad \frac{2}{\sqrt{2}}} \\
 \eta^{(1)} \quad \boxed{1 \quad -1} \quad \longrightarrow \quad \boxed{\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}} \\
 \hline \hline
 \boxed{\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{3}{\sqrt{2}}}
 \end{array}$$

$$\begin{array}{l}
 \xi^{(2)} \quad \boxed{\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{3}{\sqrt{2}}} \quad \longrightarrow \quad \boxed{\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{3}{2}} \\
 \eta^{(2)} \quad \boxed{\frac{1}{\sqrt{2}} \quad -\frac{5}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}} \quad \longrightarrow \quad \boxed{\frac{1}{2} \quad -\frac{1}{2} \quad -\frac{5}{2} \quad \frac{5}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}} \\
 \hline \hline
 \boxed{1 \quad 0 \quad -3 \quad 2 \quad 1 \quad 0 \quad 1 \quad 2}
 \end{array}$$

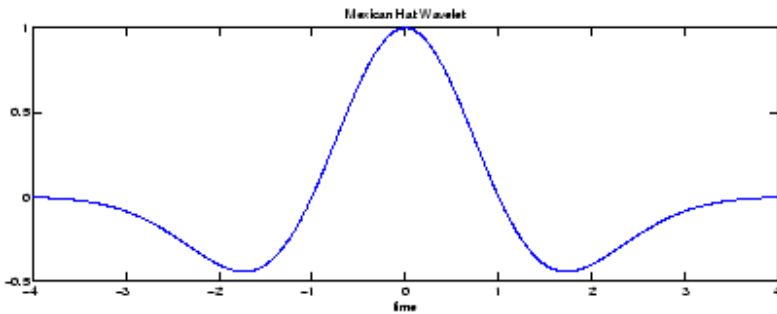




Haar waveleta

$$w(t) = \begin{cases} 1 & 0 \leq t \leq 1/2 \\ -1 & 1/2 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

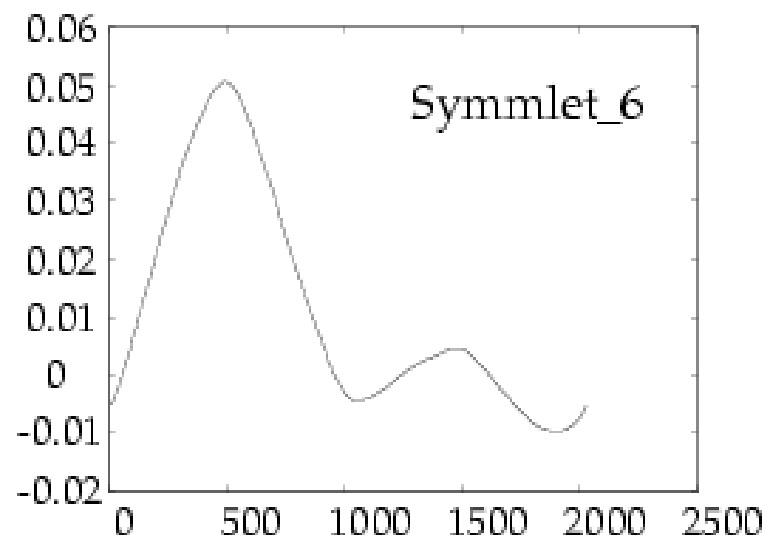
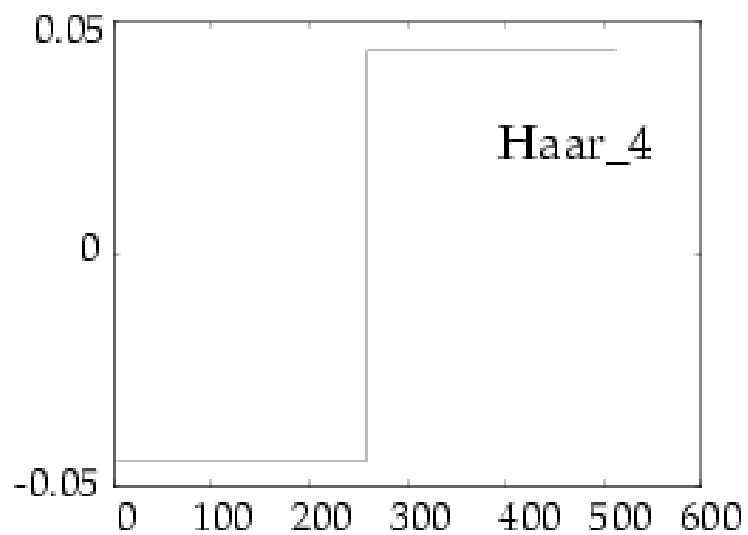
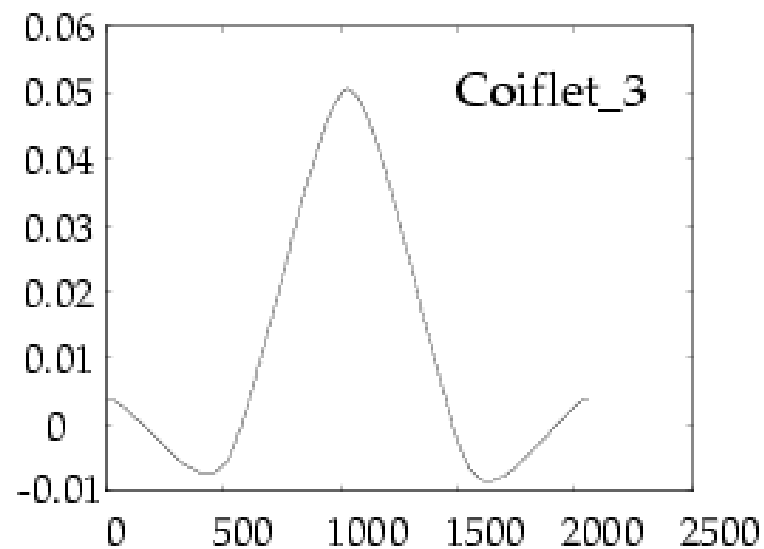
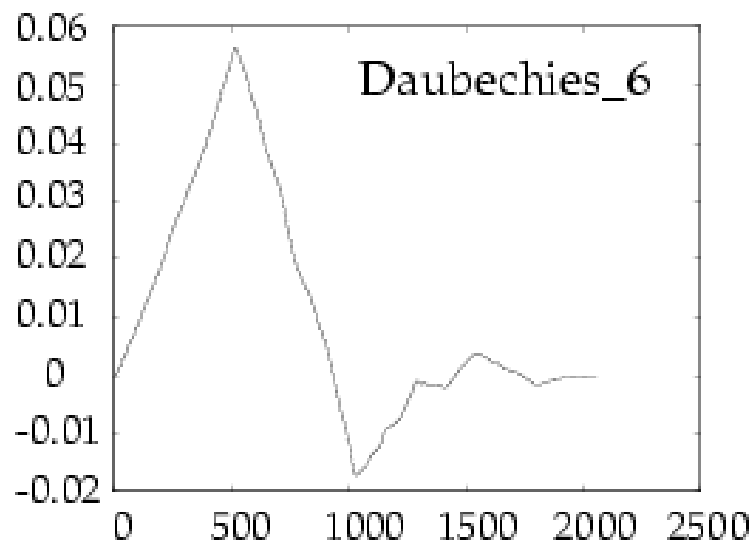
$$W(f) = j e^{-j\pi f} \frac{\sin^2(\pi f/2)}{\pi f/2}$$



Mexican hat waveleta

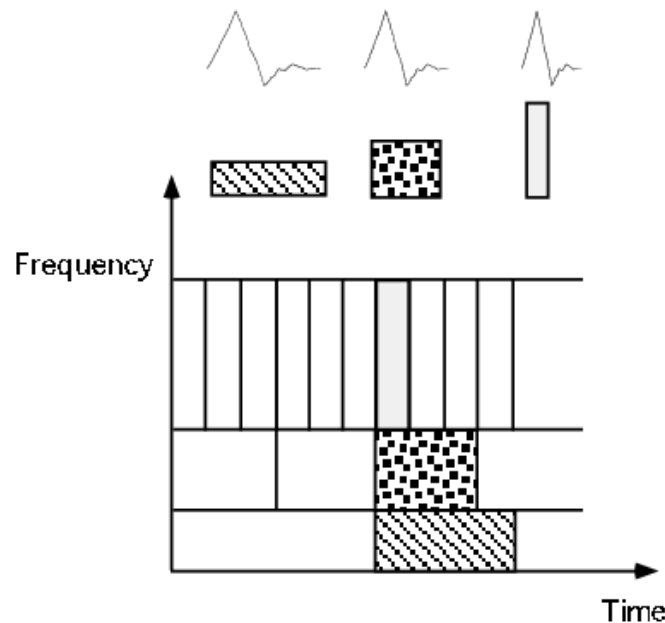
$$w(t) = (1 - t^2)e^{-t^2/2}$$

$$W(f) = \sqrt{2\pi} (2\pi f)^2 e^{-(2\pi f)^2/2}$$



Wavelet transformace

- Okno proměnné šířky
 - analýza vysokých frekvencí → úzké okno pro lepší „time“ rozlišení
 - analýza nízkých frekvencí → širší okno pro lepší „frequency“ rozlišení



Okénková Fourierova transformace

$$F_W(\tau, f) = \int_{-\infty}^{\infty} f(t) \underbrace{w^*(t - \tau) e^{-2i\pi f t}} dt$$

$$h(t) = w(t) e^{-2i\pi f_1 t}$$

$$h_a(t) = \frac{1}{\sqrt{|a|}} h\left(\frac{t}{a}\right)$$

translace, dilatace

$$WF(\tau, a) = \frac{1}{\sqrt{|a|}} \int f(t) h^*\left(\frac{t - \tau}{a}\right) dt$$

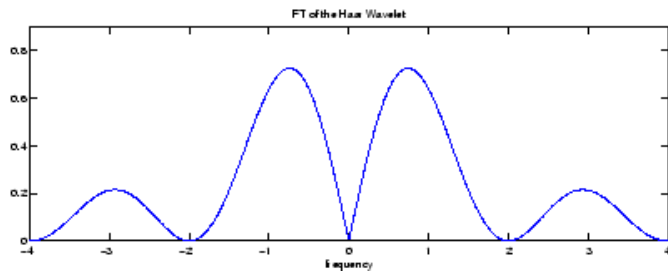
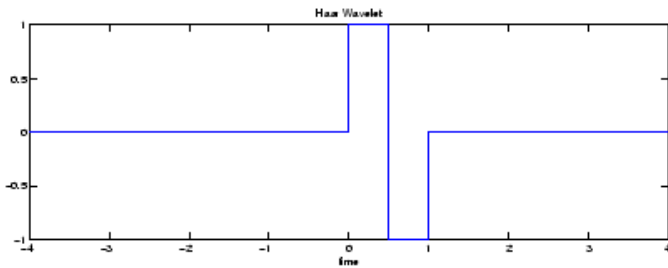
$a > 0, \in \mathbf{R}$

$\tau \in \mathbf{R}$

waveletová transformace

Waveletová transformace

$$h_{a,\tau} \Rightarrow \Psi_{a,b}$$



$$\Psi_{a,b} = \frac{1}{\sqrt{a}} \Psi\left(\frac{x-b}{a}\right)$$

$$a > 0, \in \mathbf{R}$$

$$b \in \mathbf{R},$$

normalizace přes škály

- matečná waveleta (mother wavelet)
- *wave*... osciluje
-*let* dobře lokalizovaná kolem 0, mizí rychle

$$- \int \Psi = 0$$

$$- \int |\Psi|^2 < \infty$$

$$- \text{FT}(\Psi)_{a,b} \text{ v } 0 - 0, \text{ v } \infty - 0$$

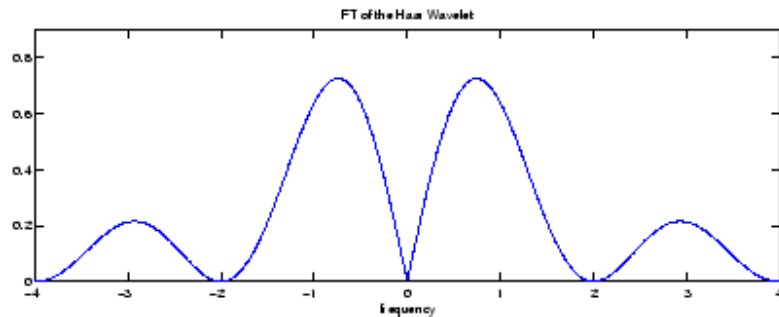
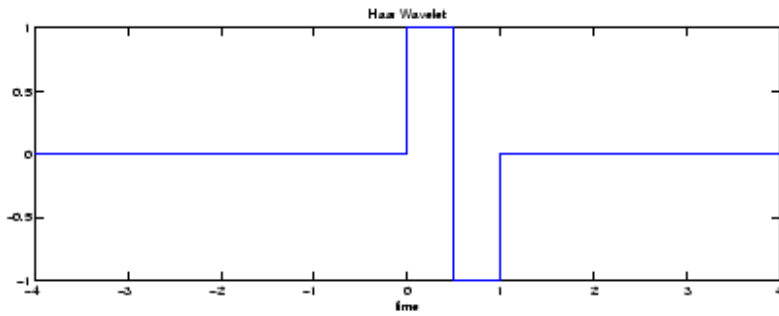
- něco jako band-pass filtr ve FT

$$\int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$

Haar waveleta

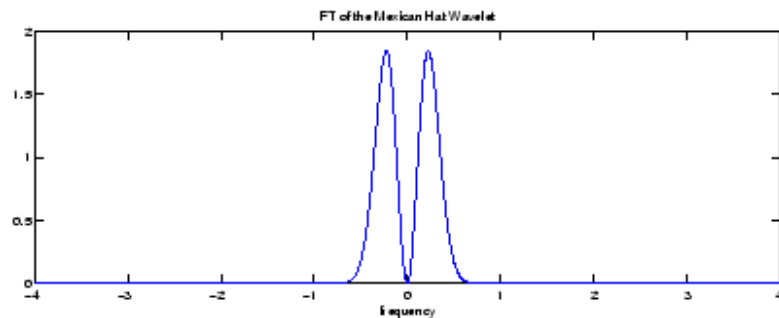
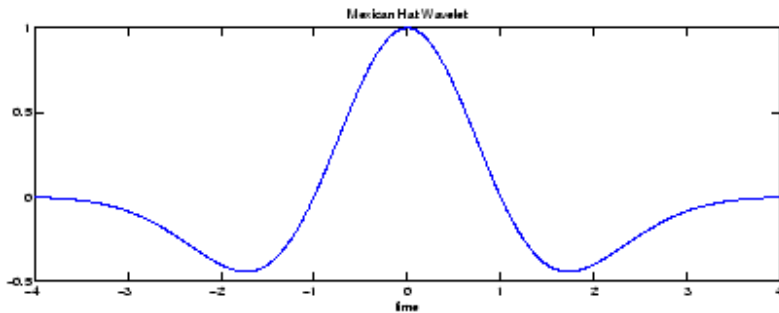
$$w(t) = \begin{cases} 1 & 0 \leq t \leq 1/2 \\ -1 & 1/2 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$W(f) = j e^{-j\pi f} \frac{\sin^2(\pi f/2)}{\pi f/2}$$



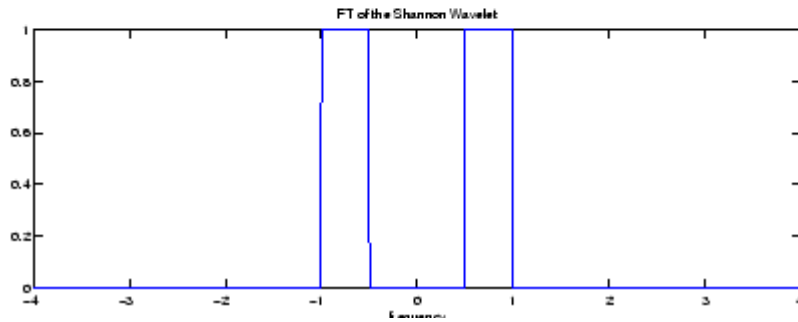
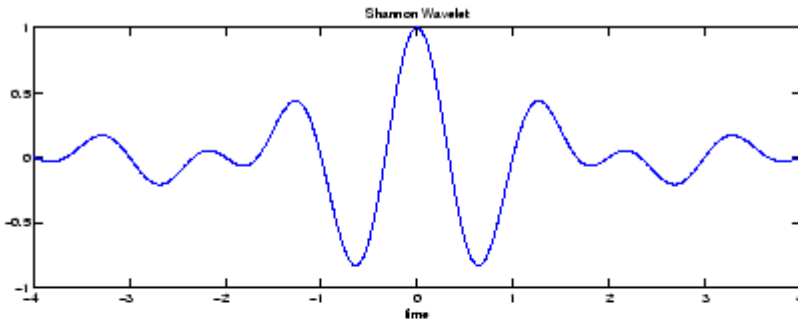
Mexican hat waveleta

$$w(t) = (1 - t^2)e^{-t^2/2} \quad W(f) = \sqrt{2\pi}(2\pi f)^2 e^{-(2\pi f)^2/2}$$



Shannon waveleta

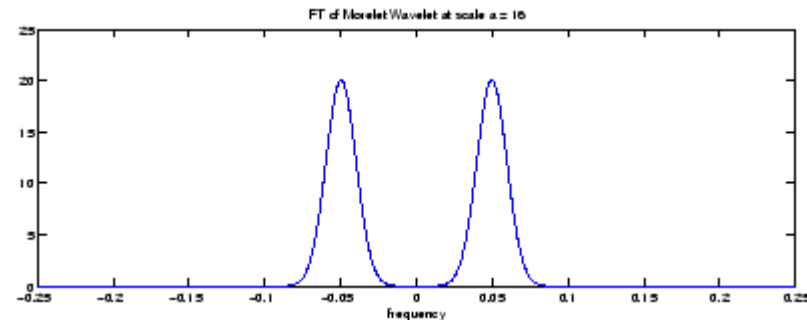
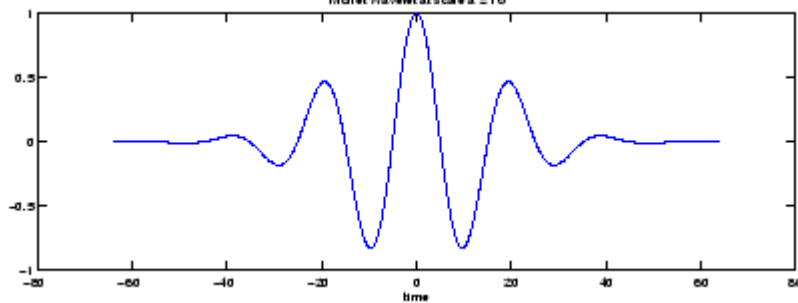
$$w(t) = \text{sinc}(t/2) \cos(3\pi t/2) \quad W(f) = \begin{cases} 1 & .5 \leq |f| \leq 1.0 \\ 0 & \text{otherwise} \end{cases}$$



Morlet waveleta

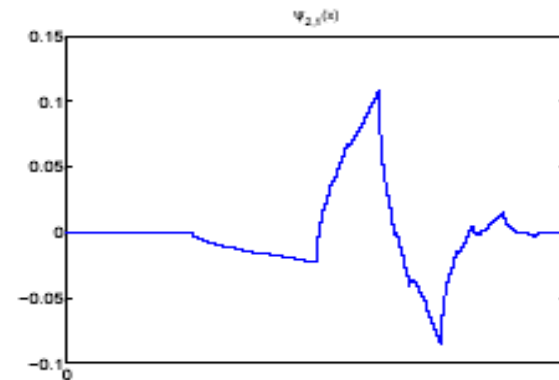
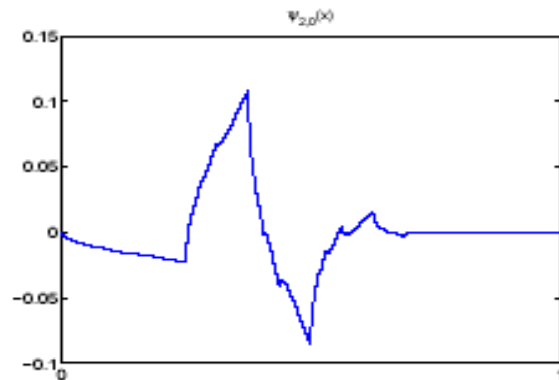
$$w(t) = e^{-t^2/2} \cos(5t)$$

$$W(f) = \sqrt{2\pi} e^{-(2\pi f - 5)^2/2} + \sqrt{2\pi} e^{-(2\pi f + 5)^2/2}$$

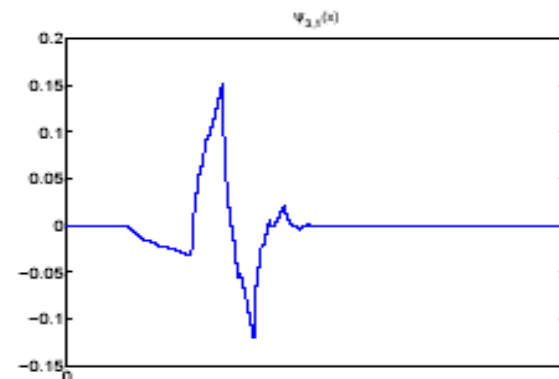
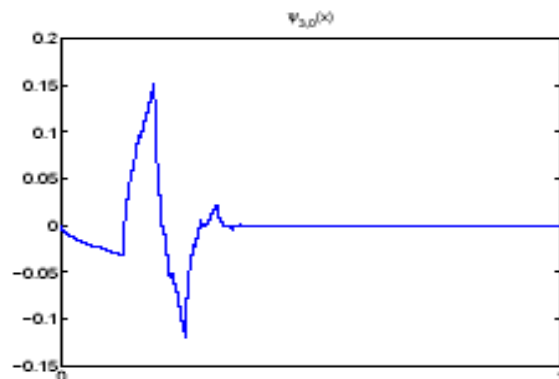


Daubechies 4 waveleta

$$\tilde{\psi}_{2,l}, l = 0, 1$$



$$\tilde{\psi}_{3,l}, l = 0, 1$$



Spojitéá waveletová transformace

$$WF(a, b) = \int f(t) \psi_{a,b}^*(t) dt$$

$$f(t) = c \int WF(a, b) \psi_{a,b}(t) \frac{da dt}{a^2}$$

c - závisí na ψ

$$WF(a, b) = \langle f(t), \psi_{a,b} \rangle$$

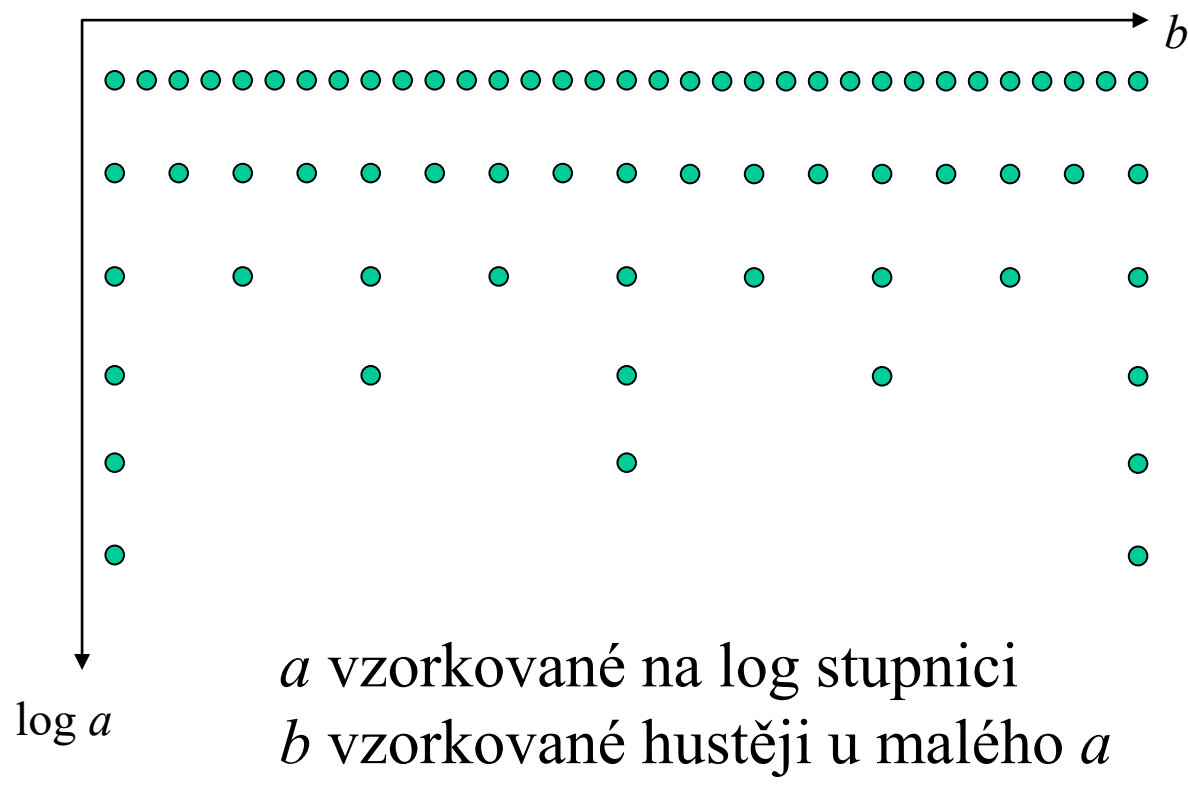
$a > 0, \in \mathbf{R}$

$b \in \mathbf{R}$

REDUNDANTNÍ!! – diskretizace a,b

Dyadická síť – diskretizace a, b

- „time“ vzorkování u nízkých frekvencí – řídké stačí



obvykle $a_0 = 2$ a $b_0 = 1$, což vede na **dyadickou síť**

Dyadická waveletová transformace

- waveletové řady

$$\psi_{m,n}(x) = \sqrt{2^m} \psi(2^m x - n).$$

binární škálování - zmenšování o faktor 2 - $-\infty < m, n < \infty$
dyadický posun - posun o $k/2^j$ $m, n \in \mathbb{Z}$

$\Psi_{m,n}$ - ortonormální báze $L^2(\mathbb{R})$

$$\langle \Psi_{m,n}, \Psi_{k,l} \rangle = \delta_{m,k} \delta_{n,l}$$

$$f(x) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_{m,n} \Psi_{m,n}$$

$$c_{m,n} = \langle f(x), \Psi_{m,n} \rangle$$

Přeurčenost

Diskrétní waveletová transformace - cesta

Kompaktní dyadická waveletová transformace

- $f(x)$, $\psi_{m,n}$ nenulové na $[0,1]$, jednotkový interval

$$\psi_j(x) = \sqrt{2^m} \psi(2^m x - n).$$

$$j = 2^m + n, m = 0, 1, \dots \quad n = 0, 1, \dots, 2^j - 1$$

pro libovolné j je m největší takové, že $2^m \leq j$, $n = j - 2^m$

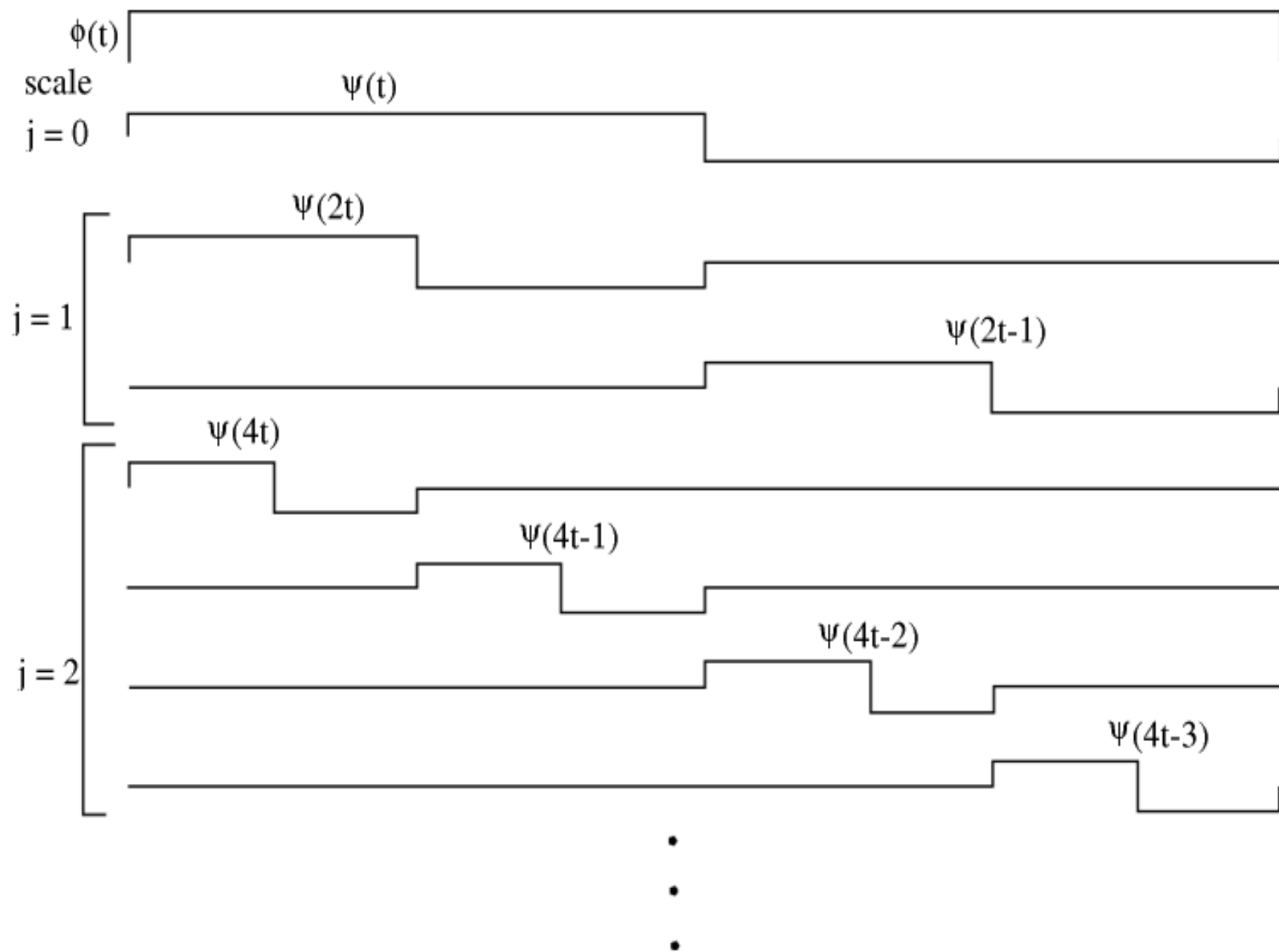
$$f(x) = \sum_{-\infty}^{\infty} c_j \psi_j$$

$$c_j = \langle f(x), \psi_j \rangle$$

Diskretizace f
N vzorků

... $f(i \Delta x)$
... mocnina 2

spojité



Diskrétní waveletová transformace

Kompaktní dyadická waveleta

Diskretizace $f \dots f(i \Delta x)$

N vzorků ... mocnina 2

$\psi_j(x)$

$$f(x) = \sum_1^N c_j \psi_j$$

$$c_j = \langle f(x), \psi_j \rangle = \sum_1^N f(x) \psi_j$$

diskrétní



FT - spojitá funkce x spojitá funkce $\int \int$

FŘ - periodická funkce x řada koeficientů $\int \Sigma$

DFT - navzorkovaná funkce x navzorkované spektrum $\Sigma \Sigma$

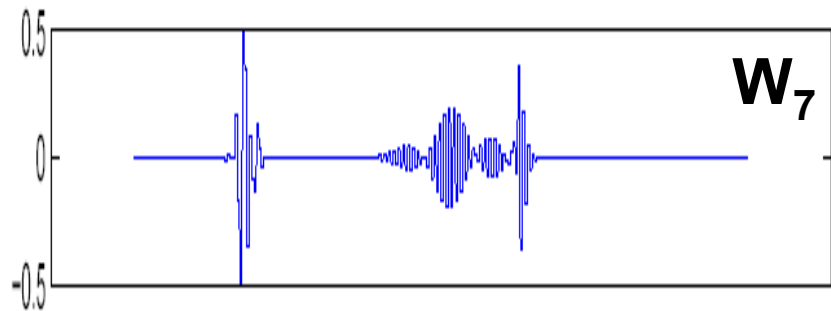
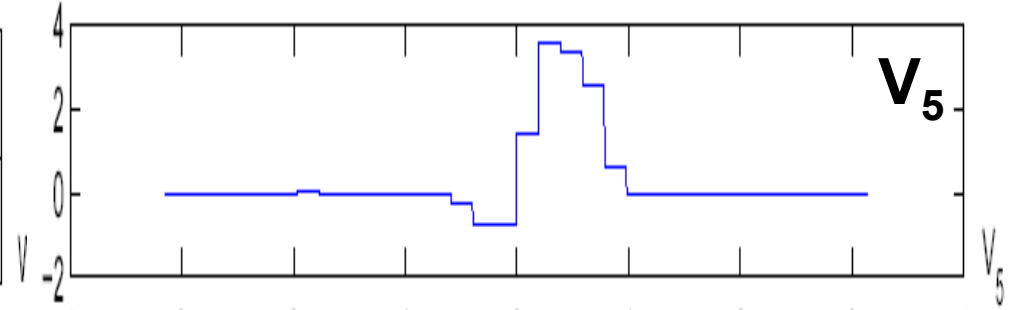
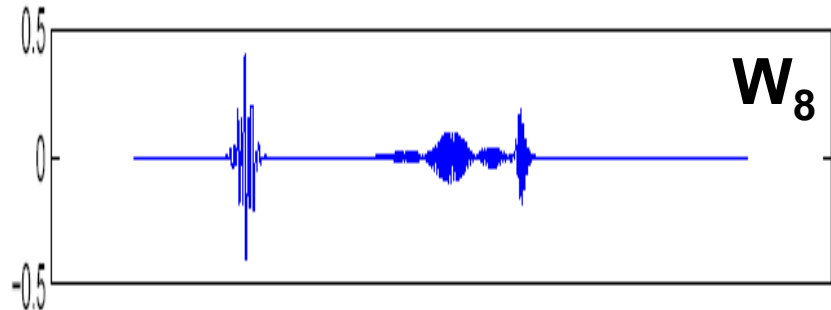
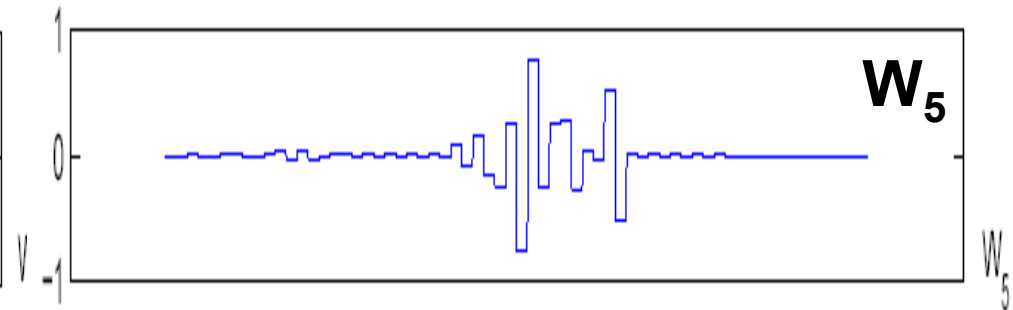
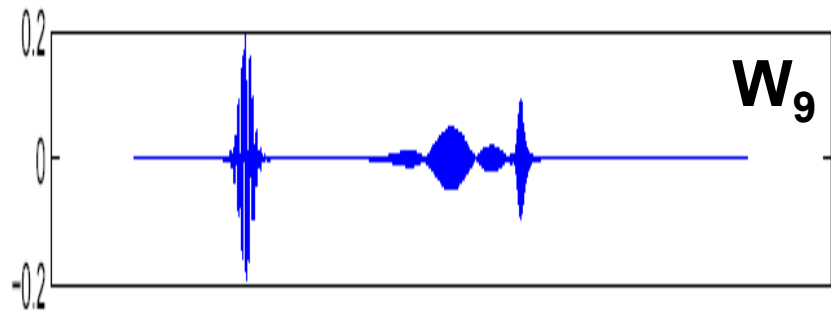
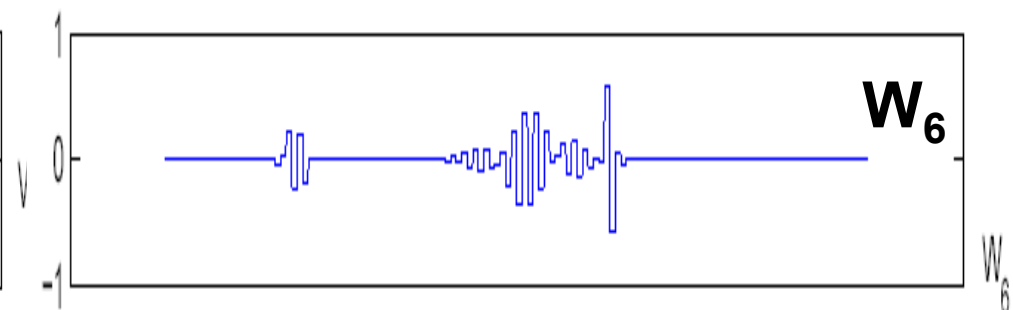
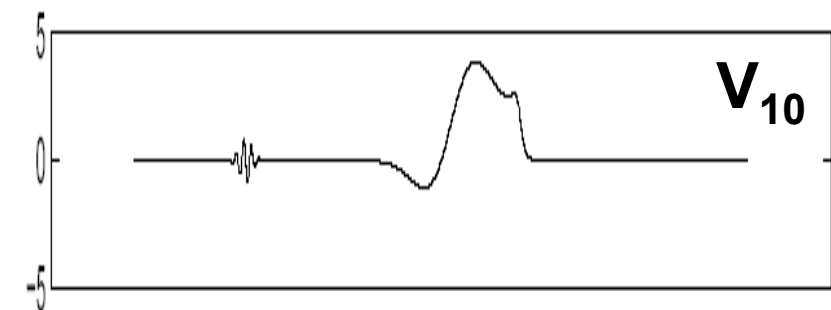
SWT - spojitá funkce x spojité a,b $\int \int$

WŘ - spojitá funkce x řada koeficientů $\int \Sigma$

DWT - navzorkovaná funkce x konečná řada koeficientů $\Sigma \Sigma$

Waveletová transformace - dekompozice

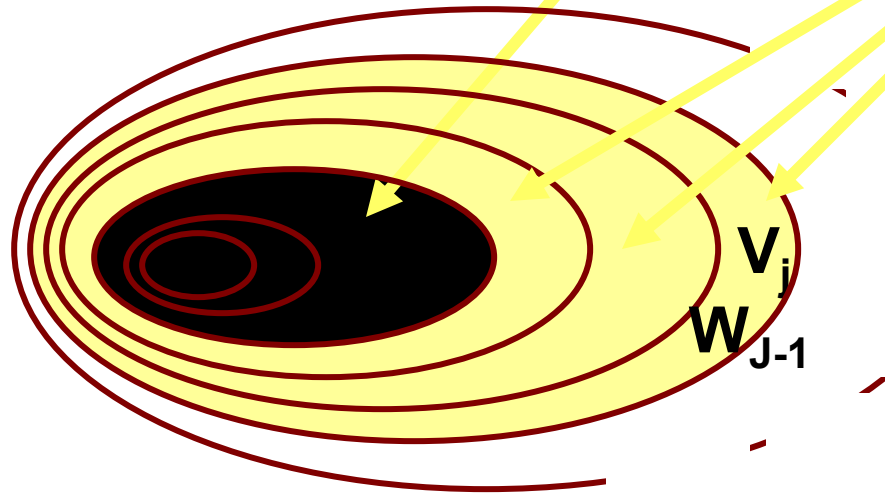




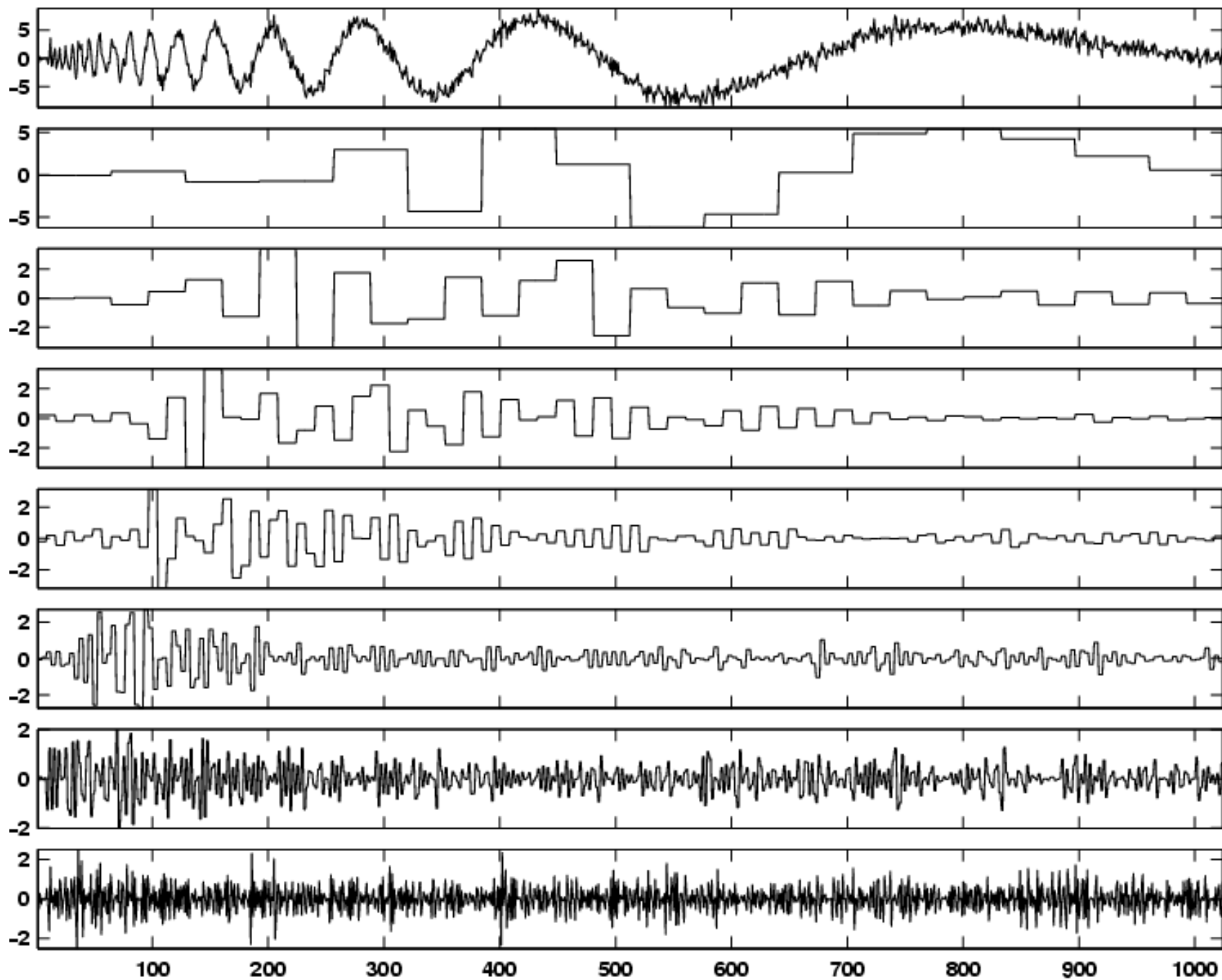
Haar waveleta

Waveletová dekompozice funkce f

$$P_{V_J} f = P_{V_{J_0}} f + \sum_{j=J_0}^{J-1} P_{W_j} f$$

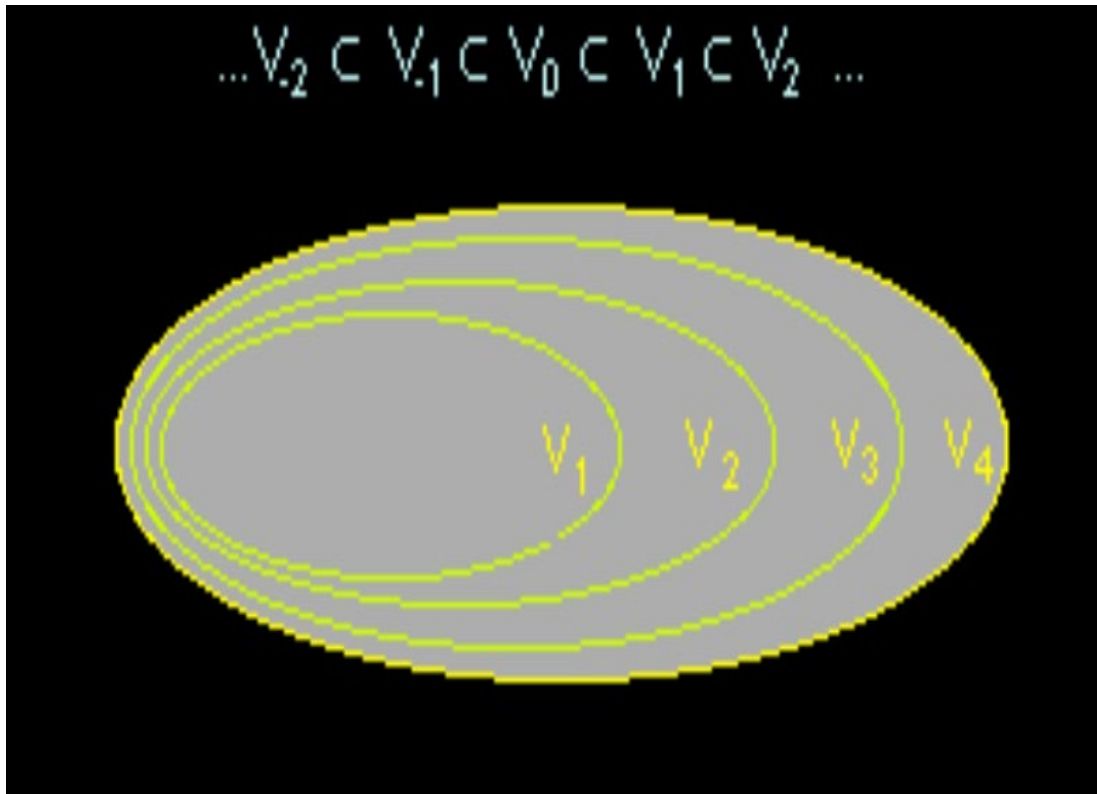


základ + \sum detaily různého měřítka



Multiresolution analysis (MRA)

- postup pro konstrukci ortonormálních bází



- L^2 prostor

- vnořená sekvence uzavřených podprostorů V_i

- každé V_i odpovídá jednomu měřítku

- plně určeno volbou škálovací funkce ϕ

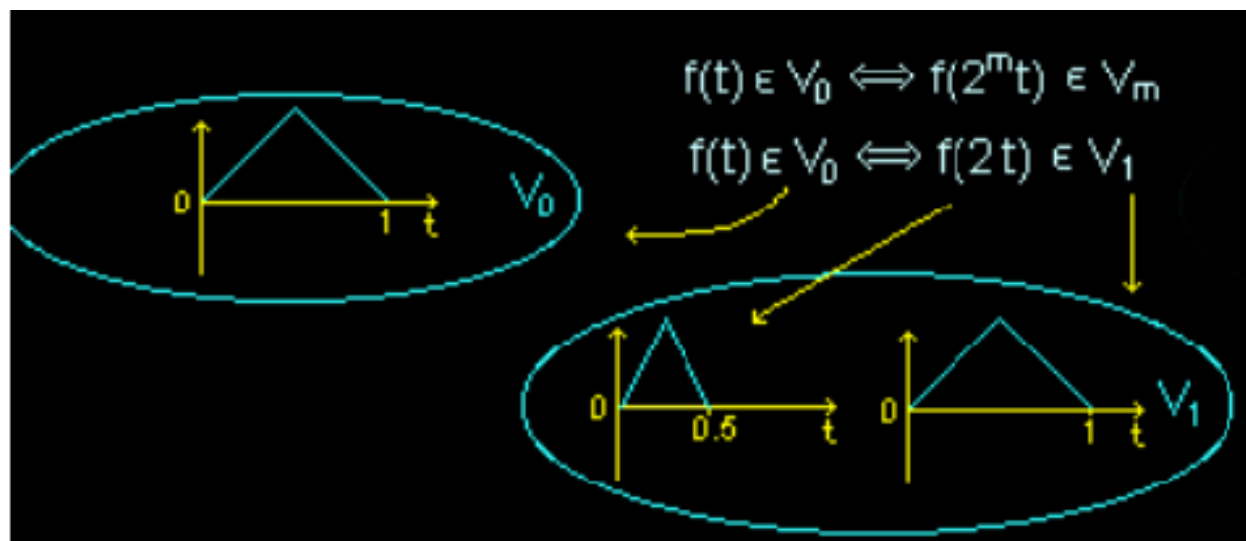
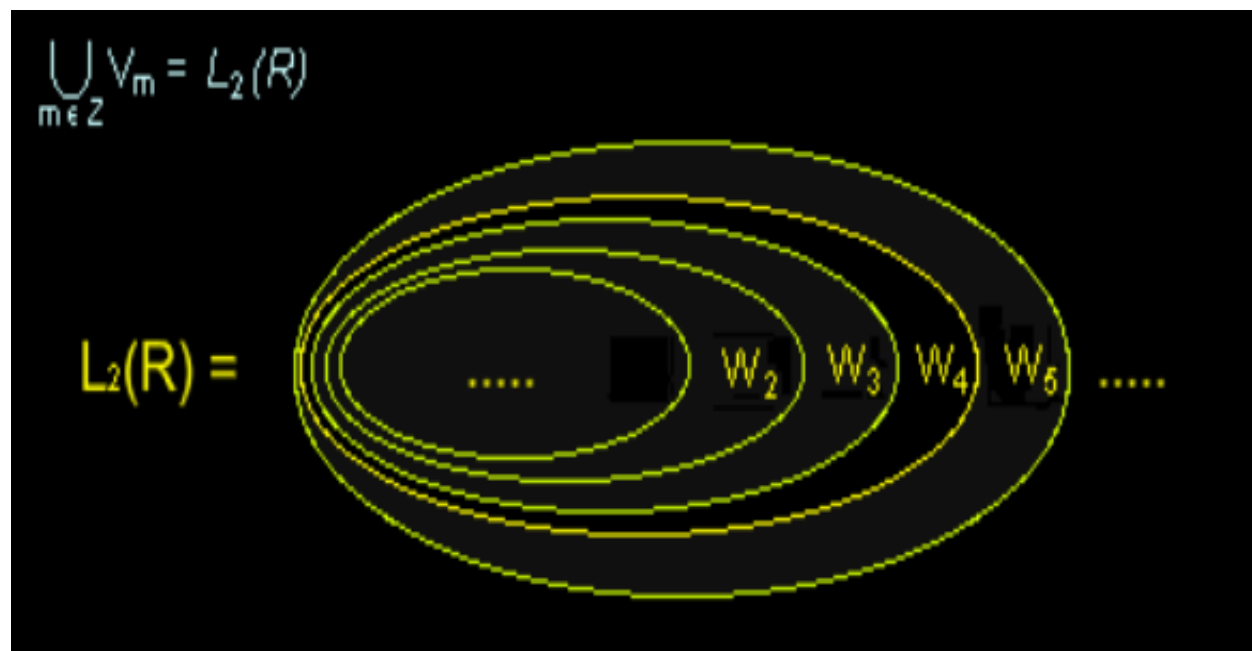
Platí:

- $\overline{\cup V_n} = L^2$

- $\cap V_n = \{0\}$

- $f \in V_0 \iff f(2^i \cdot) \in V_i$ nárůst i - jemnější rozlišení

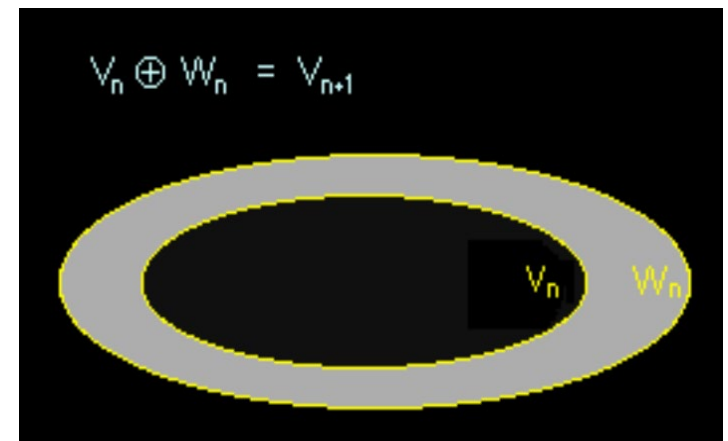
scale invariance

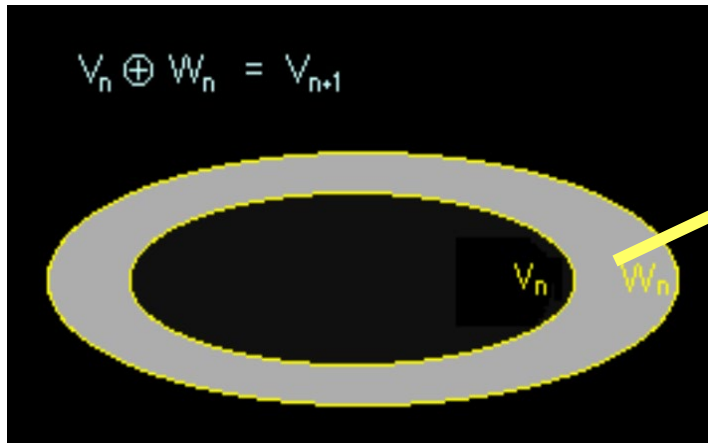


- $f \in V_i \iff f(\cdot - j) \in V_i$ **shift invariance**
- funkce $\phi_{ij}(x)$, kde $\phi_{ij}(x) = \sqrt{2^i} \phi(2^i x - j)$
tvoří ortonormální bázi V_i ... **škálovací funkce**
„father wavelet“

- $P_i(f)$ - ortonormální projekce f do V_i , pak $f = \lim_{i \rightarrow \infty} P_i(f)$.
- škálovací koeficienty $s_{ij}(f) = \langle f, \phi_{ij} \rangle$
- reprezentace chyby (detailu)
 $V_{i+1} - V_i$
- ortonormální doplněk W_i

$$W_i = V_{i+1} \ominus V_i, \quad W_i \perp V_i$$



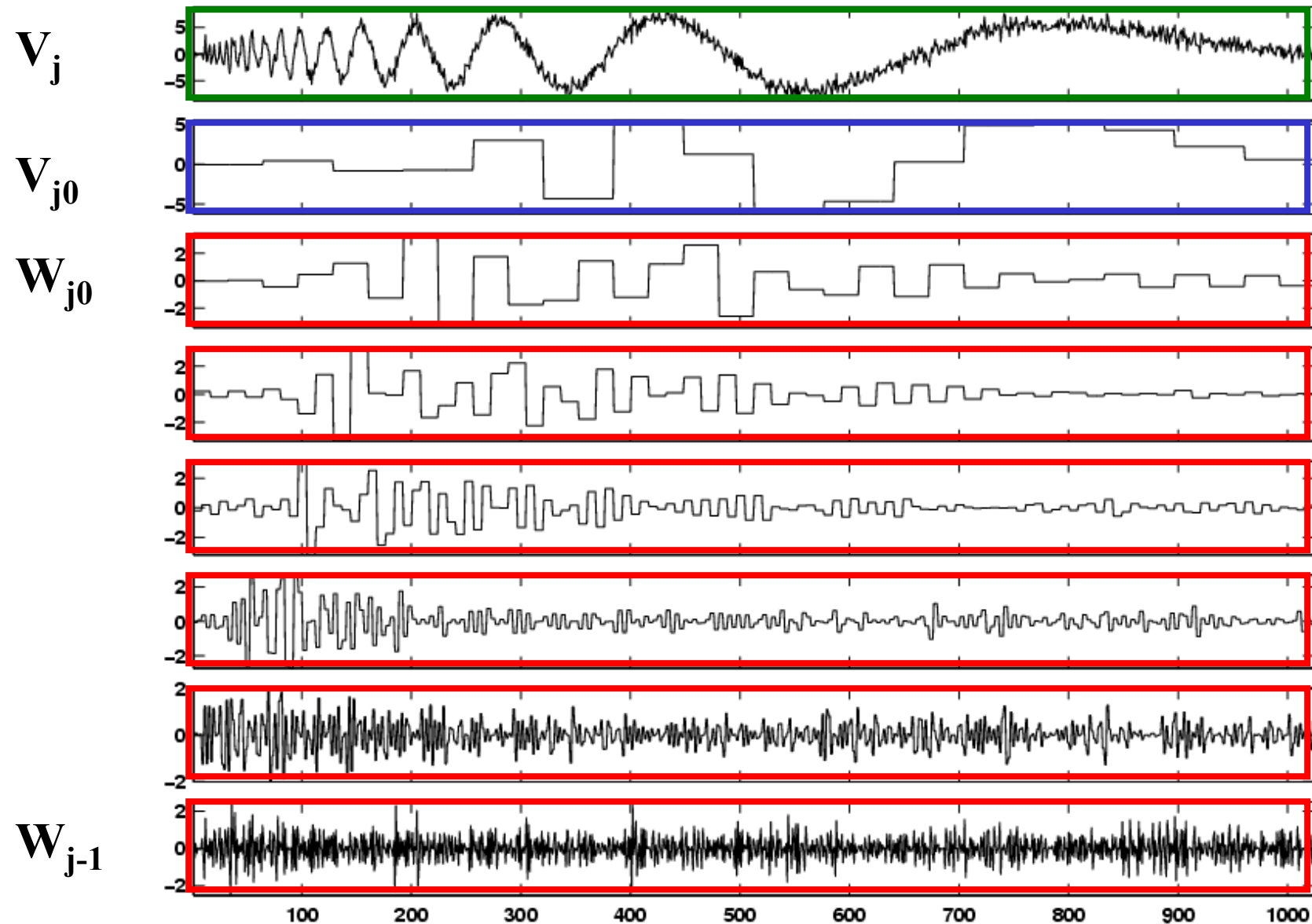


každý W_i je generován posuny $\psi_{i,j}$
waveleta

- Platí:**
- $L^2 = \bigoplus W_i$
 - škálová invariance
 - translační invariance
 - ortonormalita W_i a W_k

-
- waveletové koeficienty $w_{ij}(f) = \langle f, \psi_{ij} \rangle$

Waveletová transformace - dekompozice



● škálovací koeficienty $s_{ij}(f) = \langle f, \phi_{ij} \rangle$

● waveletové koeficienty $w_{ij}(f) = \langle f, \psi_{ij} \rangle$

● ϕ ... vyhlazovací (smoothing) funkce

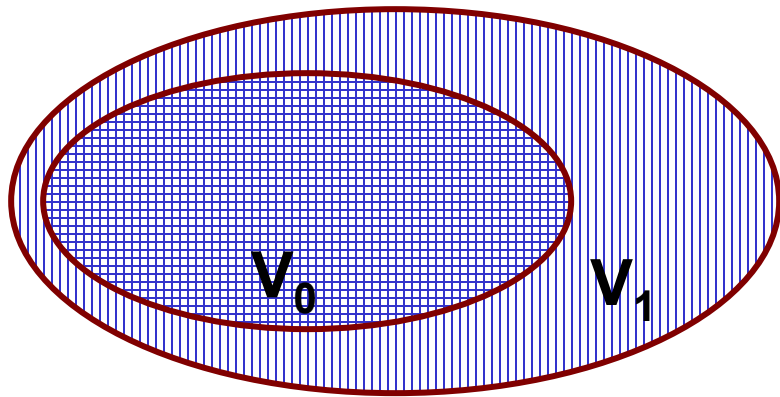
- nenulový $\int \phi (=1)$

- $\int \psi = 0$

- ϕ a $FT(\phi)$ dobrý pokles (lokalizace v obou oblastech)

- kompaktní ϕ, ψ

- nulové krom určitého konečného intervalu



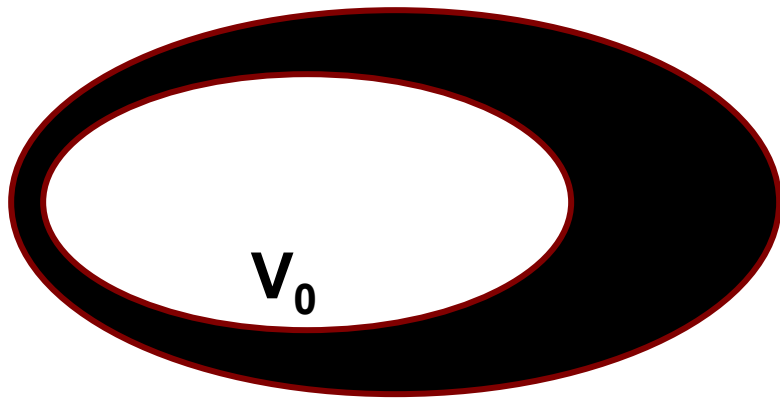
$$V_0 \subset V_1$$

$$\sum_j |h_j|^2 < \infty$$

dilatační rovnice

$$\phi(x) = \sqrt{2} \sum_j h_j \phi(2x - j)$$

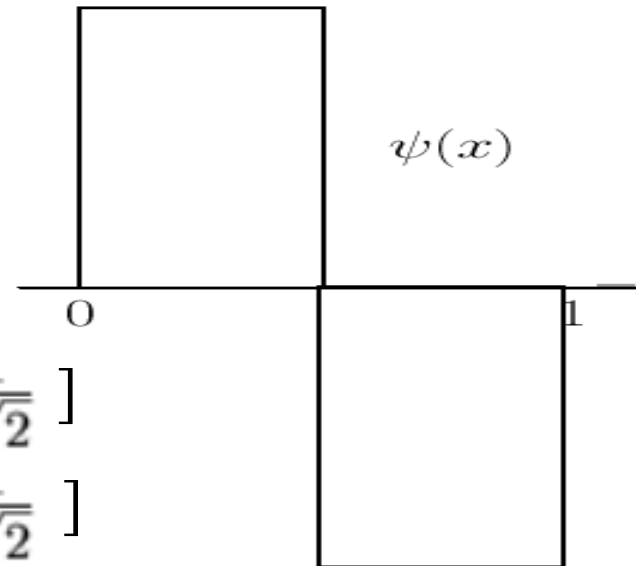
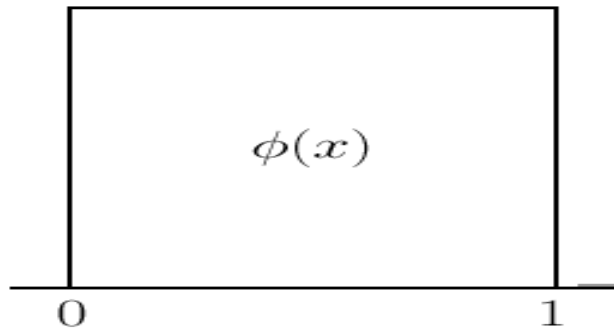
$$\psi(x) = \sqrt{2} \sum_j g_j \phi(2x - j)$$



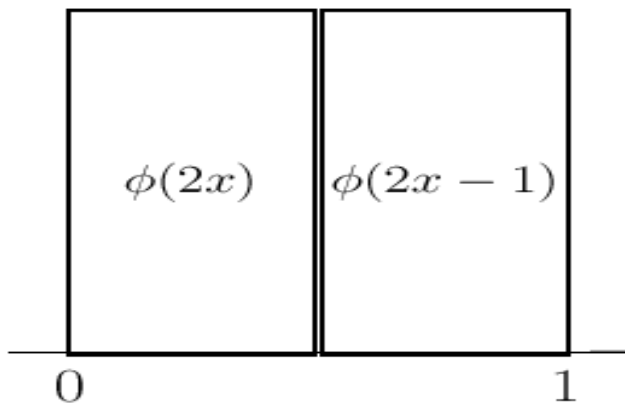
$$W_0 \subset V_1$$

V_1

Haar waveleta



$$\mathbf{h} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$$
$$\mathbf{g} = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]$$



$$\phi(x) = \phi(2x) + \phi(2x - 1)$$
$$\psi(x) = \phi(2x) - \phi(2x - 1)$$

Poznámky k h a g

h, g quadrature mirror filtry ($|H|^2 + |G|^2 = 1$)

h - low pass filtr $\sum h = \sqrt{2}$

g - high pass filtr $\sum g = 0$

g – h zpětně se změněnými znaménky
posun o pul periody

$$h = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$$

$$g = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]$$

$$h_{N-1-j} = (-1)^j g_j$$

$$g = [h_3 -h_2 h_1 -h_0]$$

h_j určuje škálovací funkci $\phi(x) = \sqrt{2} \sum_j h_j \phi(2x - j)$

$$h_k = \langle \phi_1 k, \phi \rangle = \sqrt{2} \int_{-\infty}^{\infty} \phi(t) \phi(2t - k) dt.$$

$$g_k = \langle \phi_1 k, \psi \rangle = \sqrt{2} \int_{-\infty}^{\infty} \psi(t) \phi(2t - k) dt.$$

Ortogonalita

$$\int_{-\infty}^{\infty} \phi_{j,k}(x) \phi_{j,l}(x) dx = \delta_{k,l}$$

$$\int_{-\infty}^{\infty} \psi_{i,k}(x) \psi_{j,l}(x) dx = \delta_{i,j} \delta_{k,l}$$

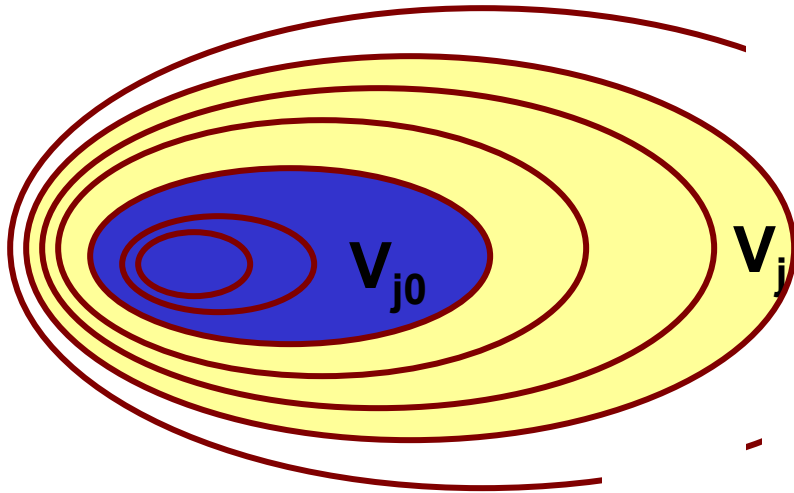
$$\int_{-\infty}^{\infty} \phi_{i,k}(x) \psi_{j,l}(x) dx = 0, \quad j \geq i$$

ψ	waveleta (wavelet)	báze W_i
ϕ	škálovací funkce (scaling function)	báze V_i

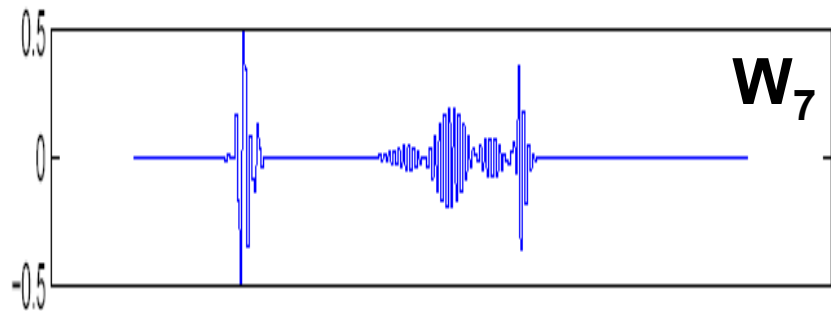
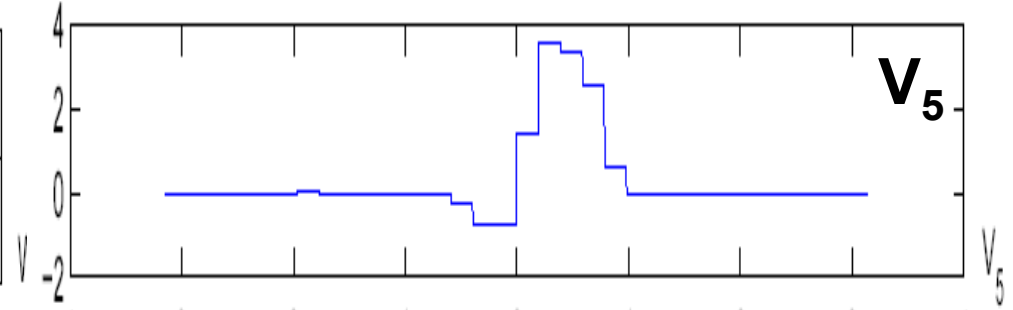
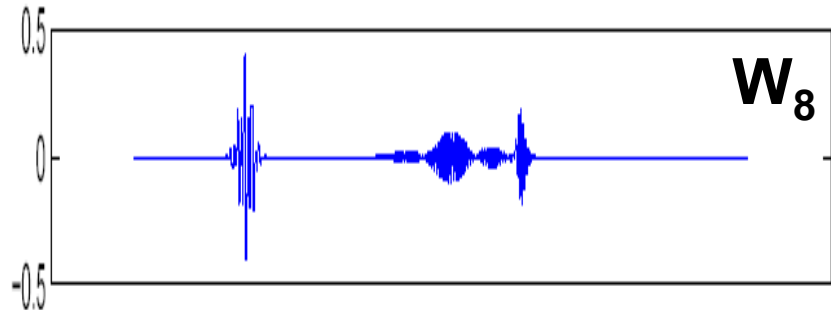
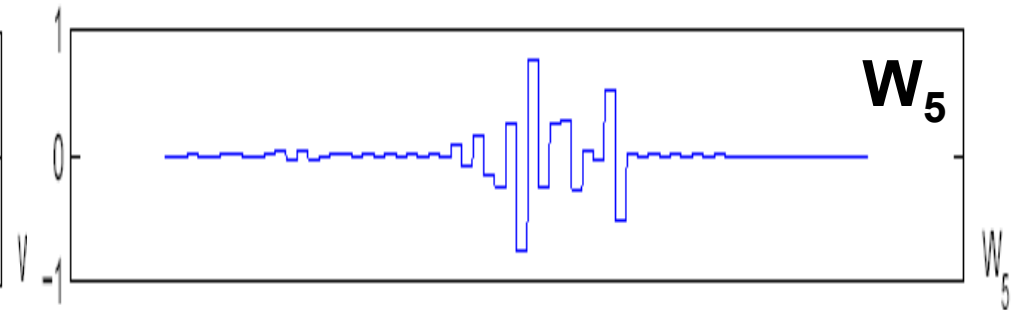
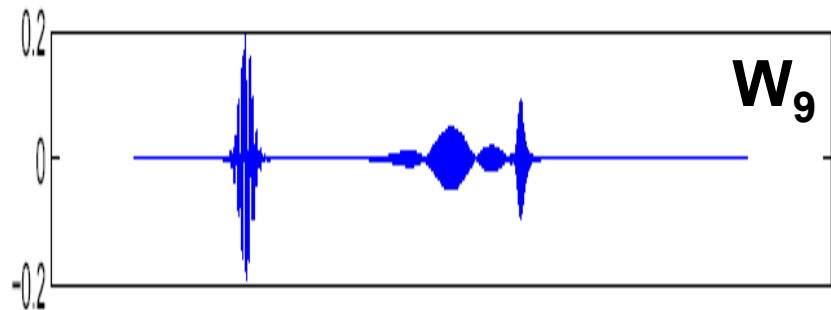
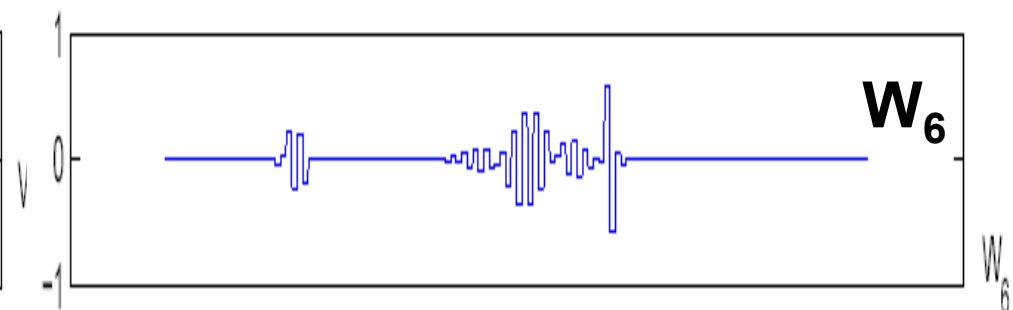
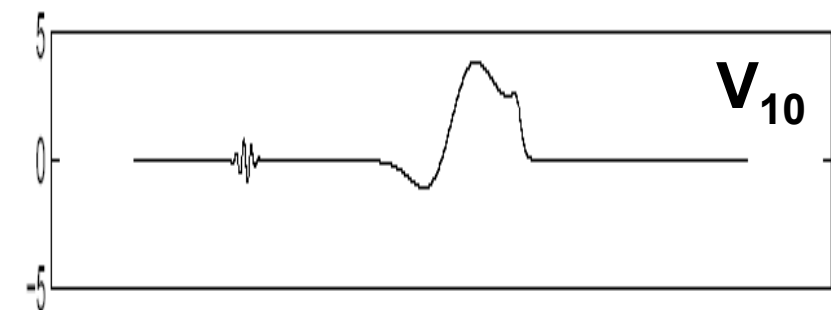
$$W_i = V_{i+1} \ominus V_i, \quad W_i \perp V_i \quad \delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Waveletová dekompozice funkce f

$$P_{V_J} f = P_{V_{J_0}} f + \sum_{j=J_0}^{J-1} P_{W_j} f$$

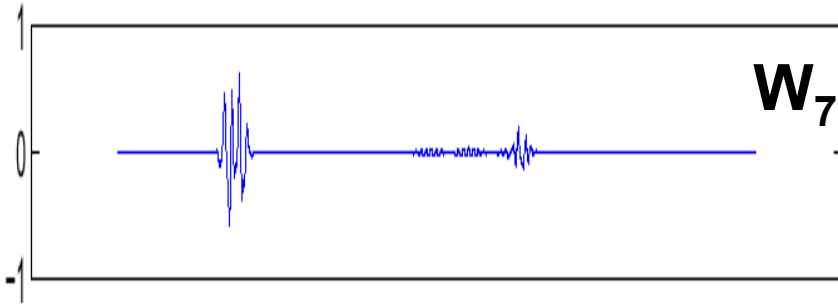
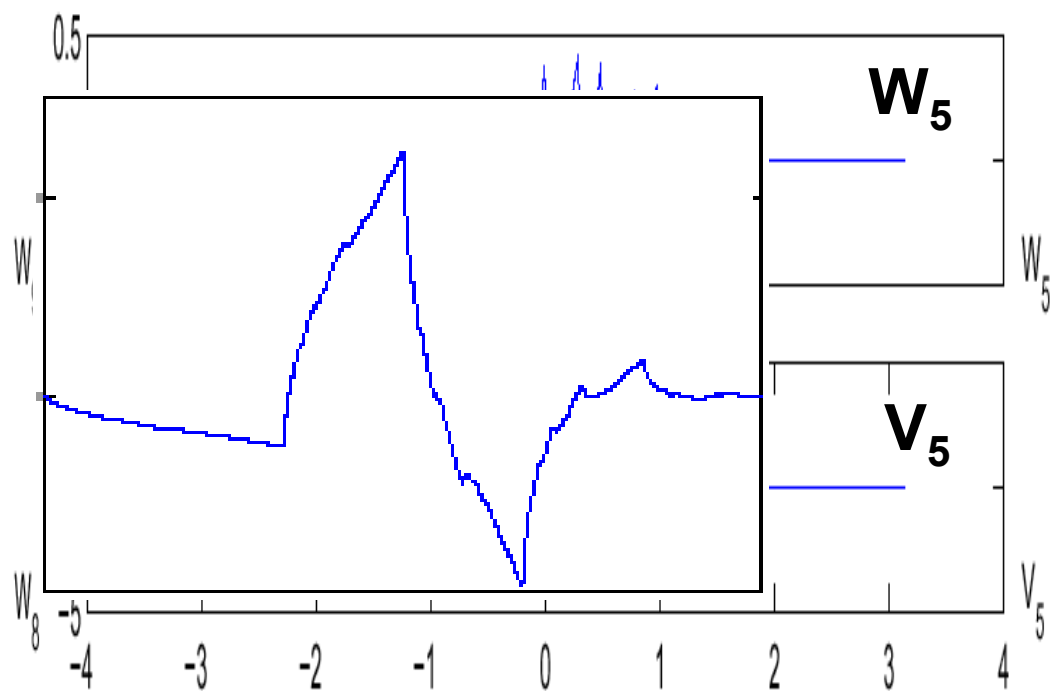
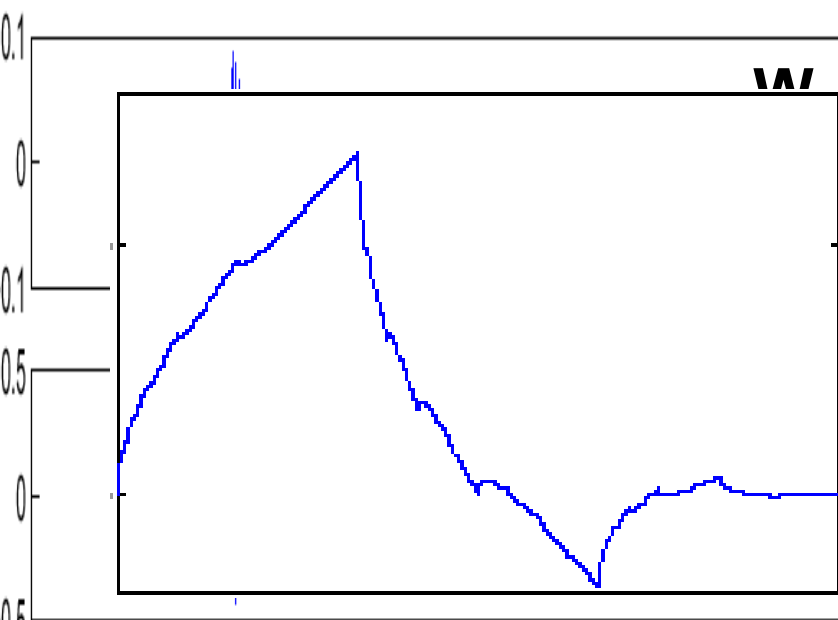
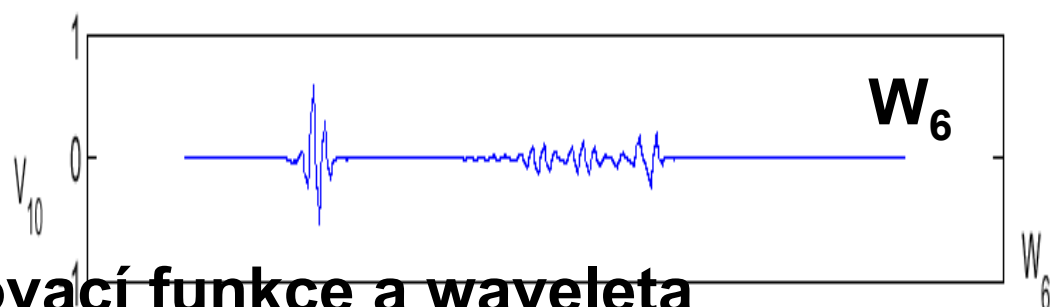


základ + \sum details různého měřítka

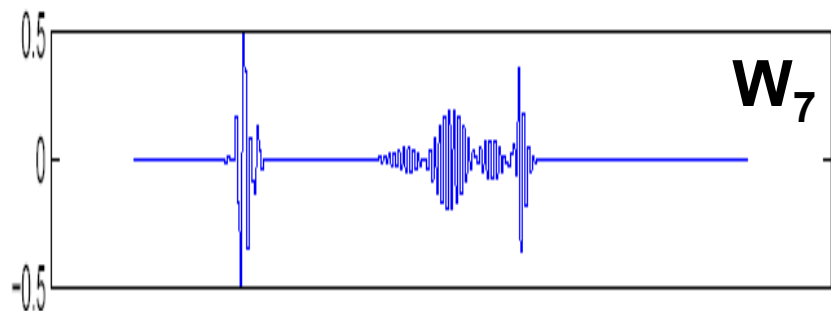
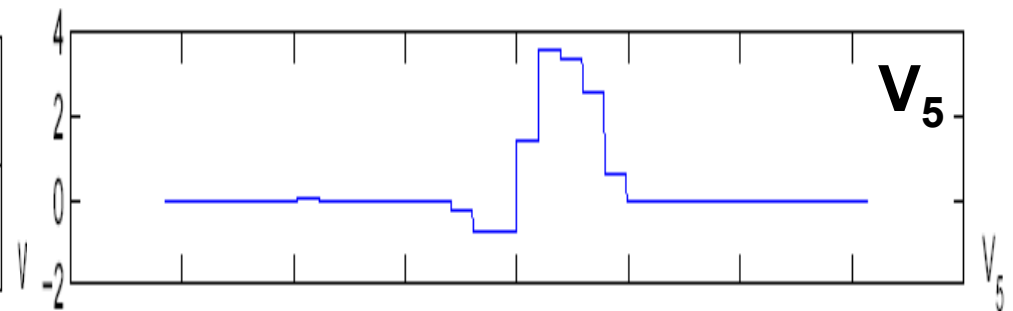
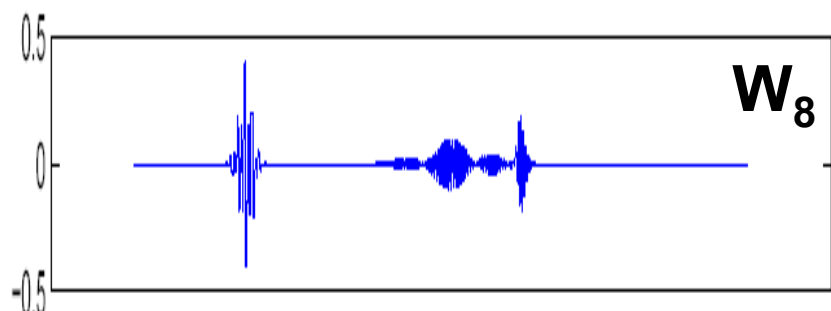
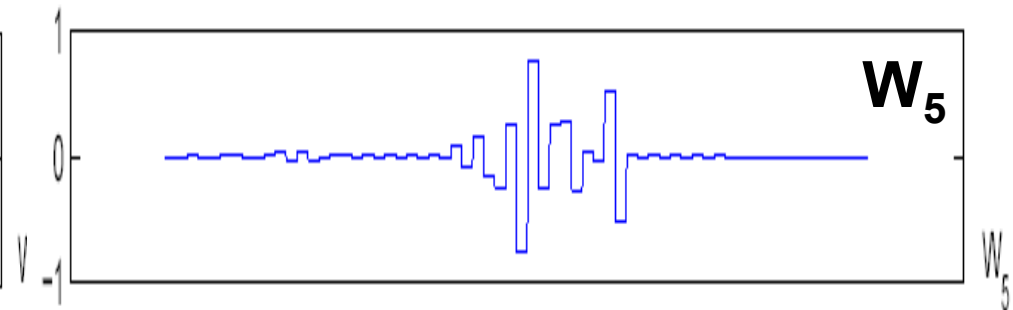
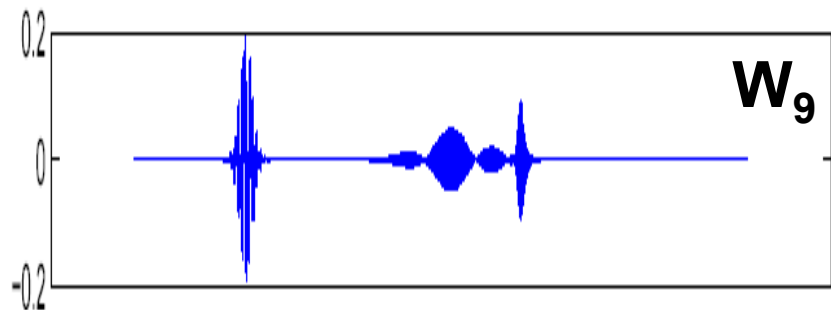
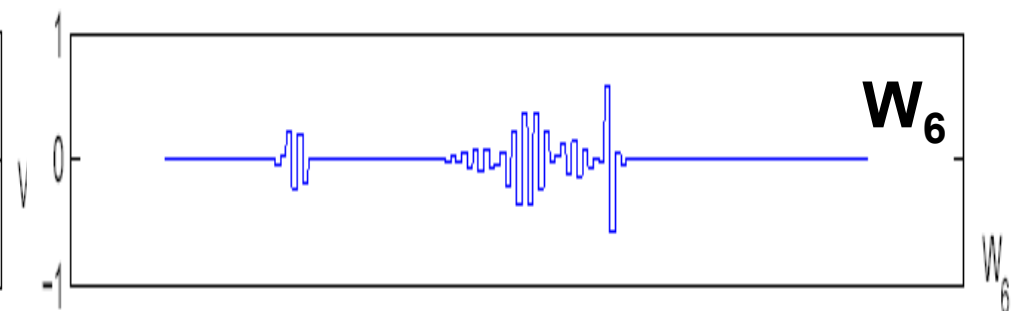
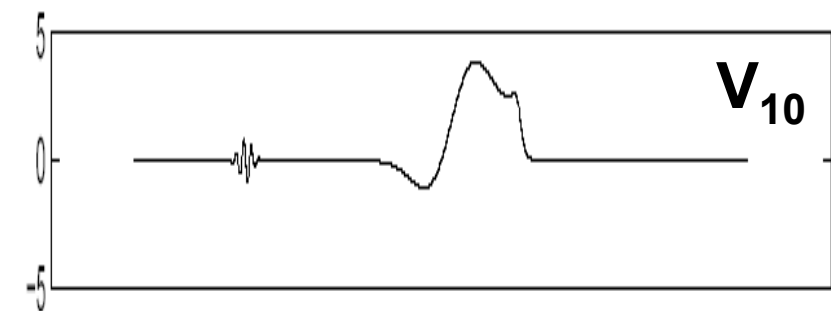


Haar waveleta

Daubechies 4 škálovací funkce a waveleta



Daubechies 4 waveleta



Haar waveleta

Waveletová dekompozice funkce f

$$P_{V_J} f = P_{V_{J_0}} f + \sum_{j=J_0}^{J-1} P_{W_j} f$$

$P_{V_j} f$ - ortonormální projekce f do V_j

$$\begin{aligned} (P_{V_j} f)(x) &= \sum_{k=0}^{2^j-1} c_{j,k} \phi_{j,k}(x) & c_{j,l} &= \int_{-\infty}^{\infty} f(x) \phi_{j,l}(x) dx \\ (P_{W_j} f)(x) &= \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(x) & d_{j,l} &= \int_{-\infty}^{\infty} f(x) \psi_{j,l}(x) dx \end{aligned}$$

kompaktní suport

$$f(x) = \underbrace{\sum_{k \in \mathbb{Z}} c_{J_0,k} \phi_{J_0,k}(x)}_{\text{základ}} + \sum_{j=J_0}^{J-1} \sum_{k \in \mathbb{Z}} d_{j,k} \psi_{j,k}(x) + \sum \text{details různého měřítka}$$

- \mathbf{V}_j $(P_{V_j} f)(x) = \sum_{k=0}^{2^j-1} c_{j,k} \phi_{j,k}(x)$

- $\mathbf{V}_{j-1} + \mathbf{W}_{j-1}$ $(P_V f)_j(x) = \sum_k c_{j-1,k} \phi_{j-1,k}(x) + \sum_k d_{j-1,k} \psi_{j-1,k}(x)$

- **DR** $\phi(x) = \sqrt{2} \sum_j h_j \phi(2x - j)$ $\psi(x) = \sqrt{2} \sum_j g_j \phi(2x - j)$

$$c_{j-1,k} = \sum_n h(n-2k) c_{j,n}$$

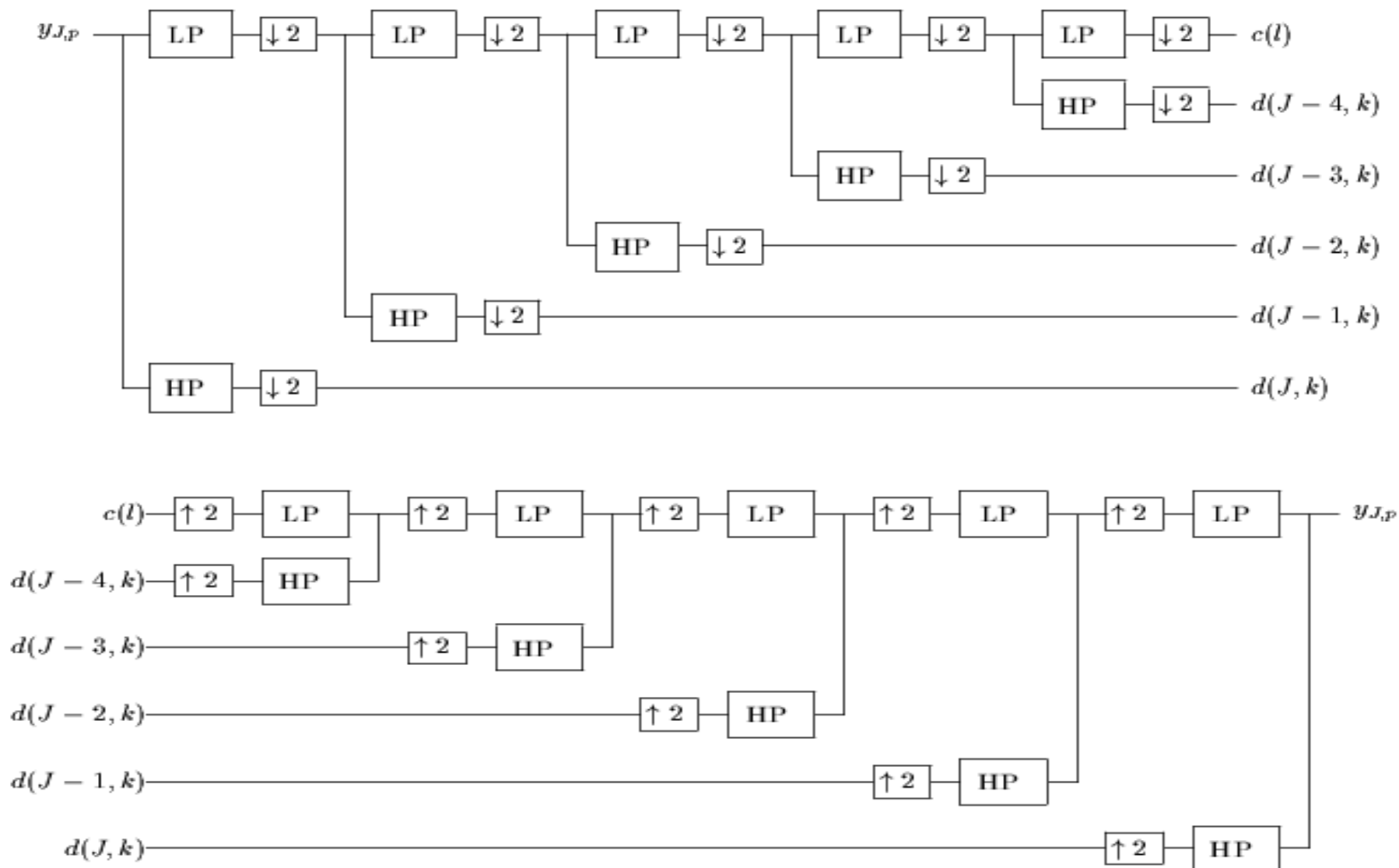
$$d_{j-1,k} = \sum_n g(n-2k) c_{j,n}$$

$$c_{j+1,k} = \sum_l h(k-2l) c_{j,l} +$$

$$+ \sum_l g(k-2l) d_{j,l}$$

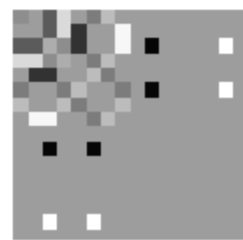
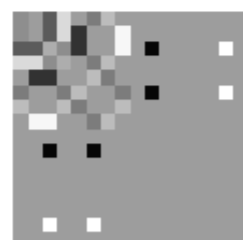
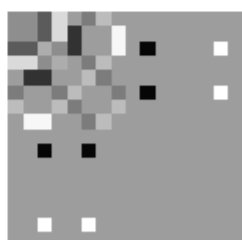
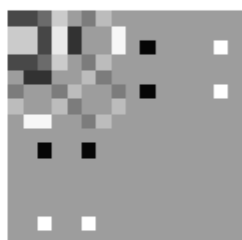
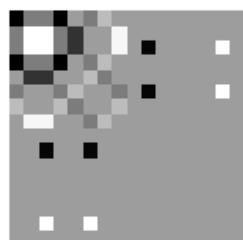
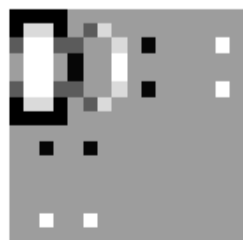
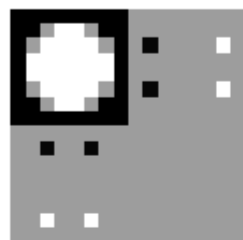
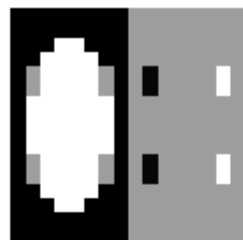
signál délky 2^J - vzorky na jednotkovém intervalu V_n
 $\langle f, \phi_{J,k} \rangle$, aproximace spojitě funkce f .. $c_{J,k}$

Rychlá waveletová transformace





$\phi_0(x)$	$\psi_0(x)$
$\phi_0(y)$	$\phi_0(y)$
$\phi_0(x)$	$\psi_0(x)$
$\psi_0(y)$	$\psi_0(y)$
$\phi_0(x)\psi_1(y)$	$\psi_0(x)\psi_1(y)$
$\phi_0(x)\psi_2(y)$	$\psi_0(x)\psi_2(y)$



$\phi_2(y)$
$\psi_2(y)$

- **Waveletová transformace - proces určení $c_{j_0,k}$, $d_{j,k}$**
- **Kompaktní - konečný počet nenulových koeficientů**
- lokalizace v čase, frekvenci
- **Požadavek na nulovost momentů**

$$0 = \int_{-\infty}^{\infty} \psi(t) dt = \dots = \int_{-\infty}^{\infty} t^{m-1} \psi(t) dt$$

- **FFT - $O(N \log_2 N)$ FWT - $O(N)$**

- **Vlastnosti očekávané od wavelet**
 - **dobrá lokalizace**
 - **jednoduchost konstrukce a reprezentace**
 - **invariance vzhledem k některým operacím**
 - **hladkost, spojitost, diferencovatelnost, symetrie**
 - **dobré vlastnosti vzhledem k počtu nulových momentů**

Kompaktnost

- v obrazové oblasti (ve frekvenční rychle k nule)
- nižší výpočetní nároky
- lepší obrazové rozlišení x horší frekvenční

Symetrie

- ortogonální kompaktní wavelety nemohou být sym.
- biortogonální wavelety

Momenty a jejich nulovost

1. M momentů 0 : signály typu $x(t) = \sum_{0 < m < M} c_m t^m$
nulové detailní koeficienty

dobré pro kompresi

Daubechies 2p koeficientů – p nulových momentů

Hladkost

lepší rekonstrukce

- **Biortogonální wavelety**

- Haar jediná kompaktní, ortogonální a symetrická

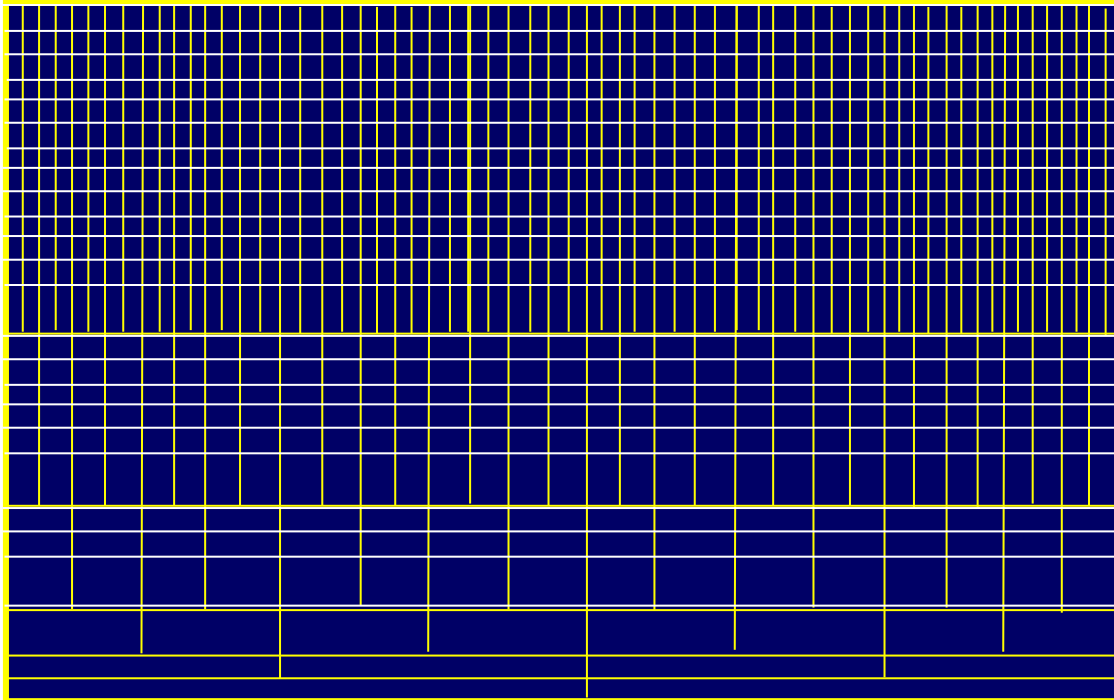
- oslabení ortogonality $\phi_{ij}, \psi_{ij}, \tilde{\phi}_{ij}, \tilde{\psi}_{ij}$

$$\langle \psi_{ij}, \tilde{\psi}_{i'j'} \rangle = \delta_{ii'} \delta_{jj'}$$

$$f = \sum_{ij} \langle f, \psi_{i,j} \rangle \tilde{\psi}_j^i = \sum_{ij} \langle f, \tilde{\psi}_j^i \rangle \psi_{i,j}$$

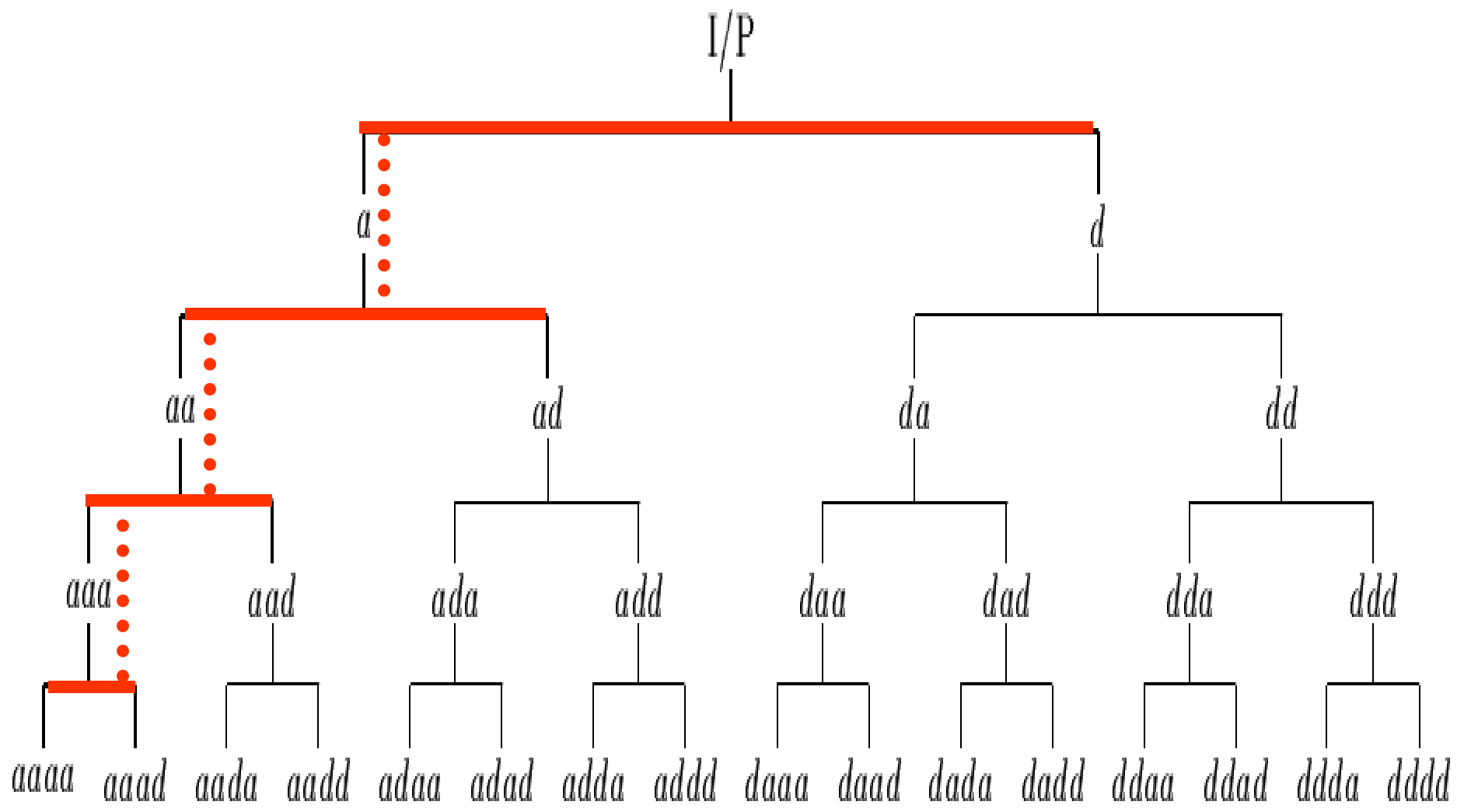
-
- **Reálné x komplexní wavelety**
 - **Ortogonální x biortogonální x neortogonální**
 - **Jiné typy diskretizace, nedyadické, m-bands**

- **Wavelet packets - nadmnožina WT**



analytické funkce (W_0 škálovací f., W_1 waveleta)

volba stromu



Filter banks

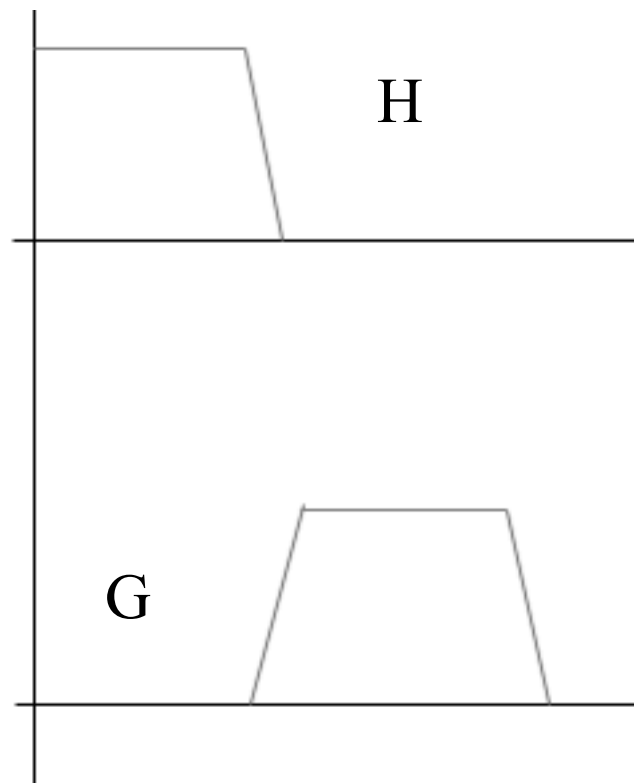
$$\psi_a(x) = (1/\sqrt{a}) \psi(x/a)$$

$$\underline{\psi}_a(x) = \psi_a^*(-x) = (1/\sqrt{a}) \psi^*(-x/a)$$

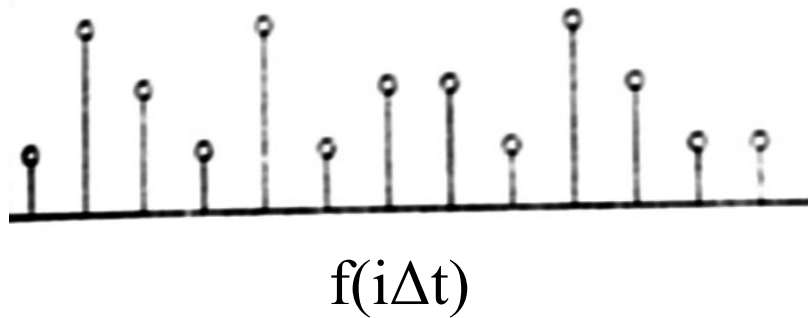
pak

$$\text{CWT} = f * \underline{\psi}_a(x)$$

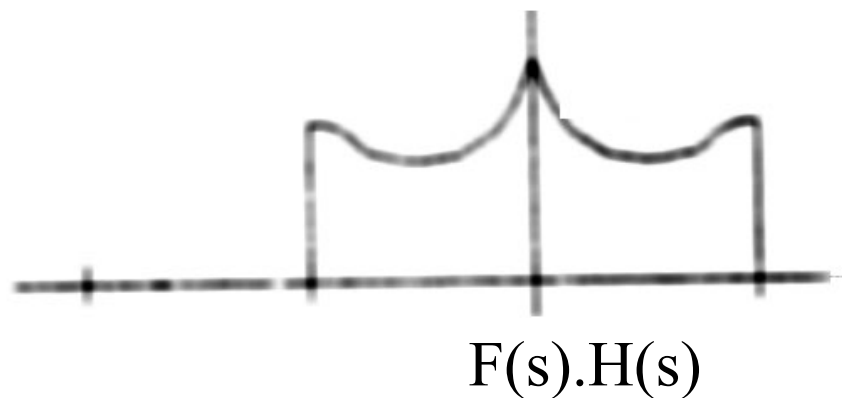
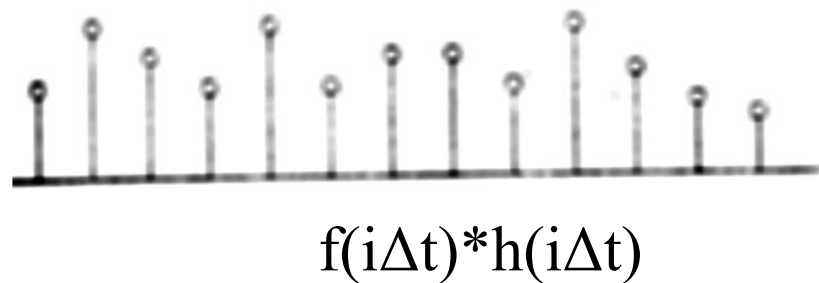
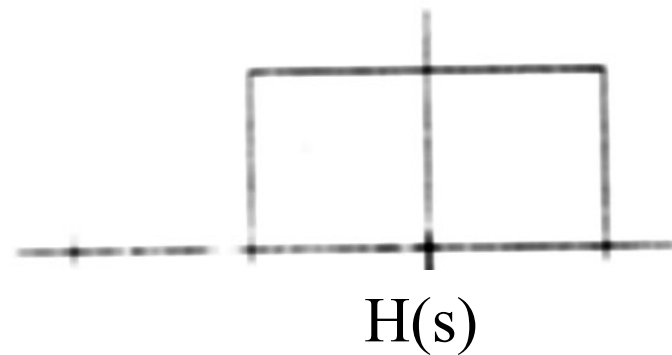
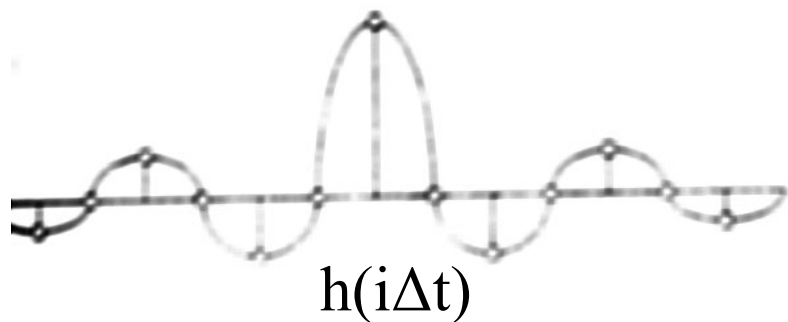
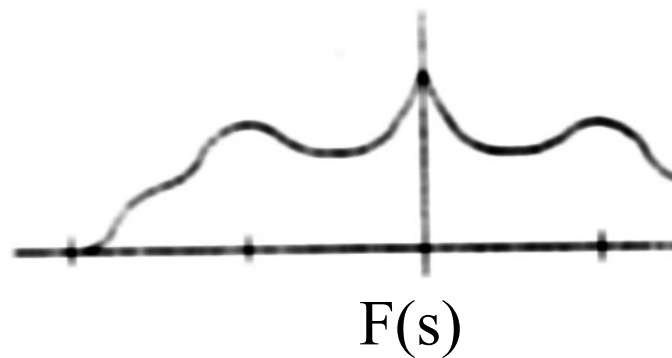
násobení ve FT



Subband coding

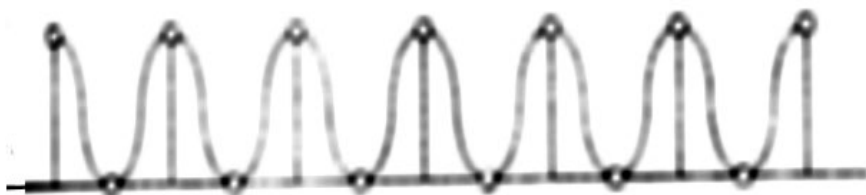


h H





$f(i\Delta t) * h(i\Delta t)$



$b(i\Delta t)$



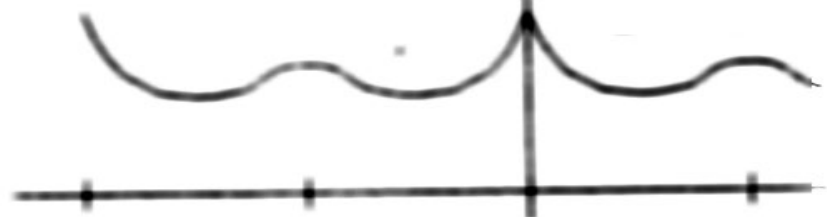
$b(i\Delta t) [f(i\Delta t) * h(i\Delta t)]$



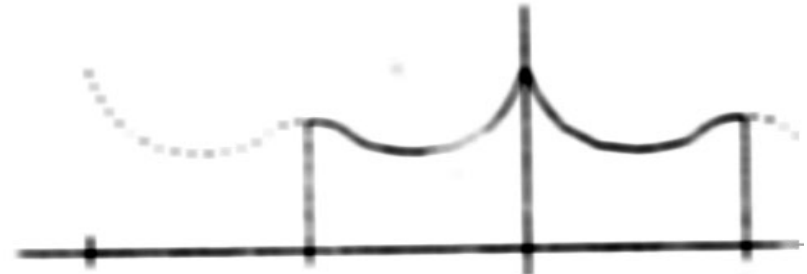
$F(s).H(s)$

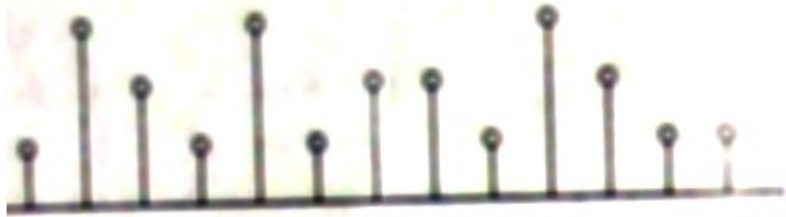


$B(s)$

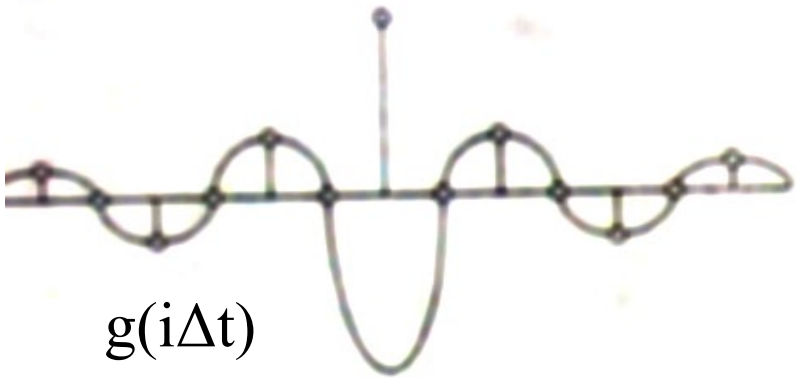


$B(s) * [F(s).H(s)]$





$f(i\Delta t)$



$g(i\Delta t)$

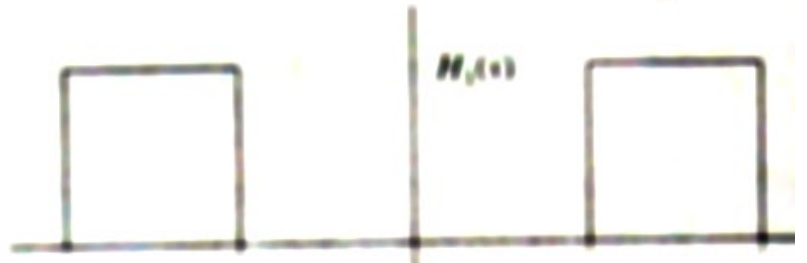


$f(i\Delta t)*g(i\Delta t)$

g G



$F(s)$



$G(s)$



$F(s).G(s)$

